Multi-D Wavelet Filter Bank Design using Quillen-Suslin Theorem for Laurent Polynomials

Youngmi Hur, Hyungju Park and Fang Zheng

***Abstract*—In this paper we present a new approach for constructing the wavelet filter bank. Our approach enables constructing nonseparable multidimensional non-redundant wavelet filter banks with FIR filters using the Quillen-Suslin Theorem for Laurent polynomials. Our construction method presents some advantages over the traditional methods of multidimensional wavelet filter bank design. First, it works for any spatial dimension and for any sampling matrix. Second, it does not require the initial lowpass filters to satisfy any additional assumption such as interpolatory condition. Third, it provides an algorithm for constructing a wavelet filter bank from a single lowpass filter so that its vanishing moments are at least as many as the accuracy number of the lowpass filter.**

***Index Terms*—Laurent polynomials, Multi-dimensional wavelets, Quillen-Suslin Theorem, Wavelet filter banks**

I. Introduction

The main objective of this paper is to present a new approach for constructing nonseparable multidimensional (multi-D) non-redundant wavelet filter banks (FBs). Constructing wavelet FBs is often reduced to solving a matrix equation with Laurent polynomial entries [1]. Connecting wavelet FBs with the Laurent polynomial matrices is usually done by the polyphase representation [2]. The key idea for our method is to decompose the z-transform of filters using, instead of the usual polyphase representation, a special type of valid (generalized) polyphase representation [3], which we obtain from the Quillen-Suslin Theorem for Laurent polynomials. This new representation allows us to use the matrix analysis techniques that were not available for the usual polyphase representation.

Quillen-Suslin Theorem (or unimodular completion), a celebrated theorem in Algebraic Geometry, states that a unimodular matrix with polynomial entries can be completed to a square polynomial matrix of determinant 1. This result was extended by R. G. Swan to unimodular matrices with Laurent polynomial entries [4].

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While there have been several uses of unimodular completion in constructing multi-D FBs [5]–[9], many of them do not provide a constructive proof for designing wavelet FBs with general sampling matrices. Furthermore, they do not offer an effective way of guaranteeing vanishing moments of the resulting wavelet systems. Our method is different from these existing methods in that it gives an algorithm to construct multi-D wavelet FBs more readily. Our method provides an algorithm for constructing a wavelet FB from a single lowpass filter so that its vanishing moments are at least as many as the accuracy number of the lowpass filter.

The wavelet representation, along with Fourier representation, has been one of the most effective data representations. Constructing 1-D wavelets is well understood by now, but the situation is quite different for multi-D case. The most commonly used method for constructing multi-D wavelets is the tensor product, but the resulting wavelets have many unavoidable limitations. For instance, tensor product multi-D wavelet FBs have large supports, and the tensor product has directional preference only along the coordinate directions.

Many researches on constructing non-tensor-based multi-D wavelet FBs or wavelets have been performed [10]–[28]. Drawbacks of existing non-tensor-based multi-D wavelet constructions include the following. Many of the existing methods work only for low spatial dimensions and cannot be easily extended to higher dimensions. Others assume that the lowpass filters or refinable functions satisfy additional conditions such as interpolatory condition.

Our construction method presents some advantages over the existing (both the tensor product and non-tenor-based) methods of multi-D wavelet construction. It works for any spatial dimension and for any sampling matrix. Furthermore, it does not require the initial lowpass filters to satisfy any additional assumption such as interpolatory condition. Being a non-tenosr-based method that works under the general setting, our method has the potential to provide filters with smaller support or more flexible directional features (cf. Example 1 in Section III-B).

We now outline the rest of our paper. In Section II, we briefly review some technical background about wavelet FBs, unimodular completion and other relevant concepts. In Section III, we present our main results together with examples illustrating our findings. We summarize our results and provide outlooks in Section IV. Appendix contains some technical proofs.

II. Preliminaries

A. Wavelet filter banks and their polyphase representation

Let Λ be an $n \times n$ integer sampling or dilation matrix. By definition, this means that Λ is an integer matrix and its spectrum lies outside the closed unit disc. Throughout the paper, we use q to denote the magnitude of $\det \Lambda$, i.e. $q := |\det \Lambda|$.

A Laurent trigonometric polynomial is typically referred to as a *mask*, and a mask τ with $\tau(0) = \sqrt{q}$ and $\tau(0) = 0$ as a *refinement mask* and *wavelet mask*, respectively. It is well known that refinement masks can be used to obtain refinable functions used in wavelet construction via the cascade algorithm (or subdivision scheme) [29] and, together with wavelet masks, they can be used to construct wavelet systems in $L^2(\mathbb{R}^n)$ [30]. We recall that a filter $f : \mathbb{Z}^n \to \mathbb{R}$ is *associated with* a mask τ if τ is the Fourier transform of f. A filter f is called *lowpass* or *refinement* if

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sqrt{q},$$

and *highpass* or *wavelet* if

$$\sum_{k \in \mathbb{Z}^n} f(k) = 0.$$

In this paper we consider only the finite impulse response (FIR) filters. A FB consists of the analysis bank and the synthesis bank, which are collections of finite number, say p, of FIR filters linked by downsampling and upsampling operators, respectively, with the sampling matrix Λ [31]. We refer to a filter from the analysis bank as an analysis filter and a filter from the synthesis bank as a synthesis filter. We consider only the FBs that satisfy the perfect reconstruction condition, which implies $p \geq q$. We are interested in the FB for which each of its analysis and synthesis banks has exactly one lowpass filter and the rest of them are all highpass filters. We refer to such a FB as a *wavelet* FB. A FB is called *critically sampled* or *non-redundant* if $p = q$ and *oversampled* or *redundant* otherwise. Designing non-redundant wavelet FBs is an important problem since it leads to the construction of wavelet bases under well-understood constraints [30]–[32].

We recall that for a filter f, the number of zeros of the Fourier transform of f at $\omega = 0$ is referred to as the *number of (discrete) vanishing moments of the filter* f [17]. Thus, a filter f is highpass if and only if f has at least one vanishing moment. We say that a wavelet FB has $s \in \mathbb{N}$ vanishing moments if the minimum of all its highpass filters' vanishing moments is s.

We use Γ to denote a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^n/\Lambda\mathbb{Z}^n$ containing 0, and Γ^* to denote a complete set of representatives of the distinct cosets of $2\pi(((\Lambda^*)^{-1}\mathbb{Z}^n)/\mathbb{Z}^n)$ containing 0. Throughout this paper, for a matrix M, M^* is used to denote its conjugate transpose. We note that both the sets Γ and Γ^* have $q = |\det \Lambda|$ elements. For example, for the 2-D dyadic dilation matrix $\Lambda = 2\mathtt{I}_2$, the sets $\Gamma = \{(0,0),(1,0),(0,1),(1,1)\}$ and $\Gamma^* = \{(0,0),(\pi,0),(0,\pi),(\pi,\pi)\}$ can be used. We also use the notation

$$\nu_0 = 0, \nu_1, \cdots, \nu_{q-1}$$

to denote the elements of Γ.

The concept of polyphase decomposition is to transform a filter or a signal into q filters or signals running at the sampling rate $1/q$ [2]. For a given FB, let h be an analysis filter, and g a synthesis filter. Then the polyphase decomposition of h (respectively, g) is a set of q filters h_ν, $\nu \in \Gamma$, (respectively, g_ν, $\nu \in \Gamma$) that are defined as

$$h_\nu(m) := h(\Lambda m - \nu), \quad g_\nu(m) := g(\Lambda m + \nu), \quad \forall m \in \mathbb{Z}^n.$$

The z-transform ([33]) $Y(z)$ of a filter $y : \mathbb{Z}^n \to \mathbb{R}$ is defined as

$$Y(z) := \mathcal{Z}\{y\} := \sum_{m \in \mathbb{Z}^n} y(m) z^{-m}$$

where for $z = [z_1, \ldots, z_n]^T \in \mathbb{C}^n \backslash \{0\}$ with $|z| = 1$ and $m = [m_1, \ldots, m_n]^T \in \mathbb{Z}^n$, z^m is defined to be $\prod_{j=1}^n z_j^{m_j}$. Here and below, T is used to represent the matrix transpose. We note that $Y(e^{i\omega})$, $\omega \in \mathbb{T}^n$, is the Fourier transform of y. We let $\mathtt{1} := [1, \cdots, 1]^T$ be the vector of ones. The z-transforms of the filters h and g can be written as

$$H(z) = \sum_{\nu \in \Gamma} z^{\nu} H_\nu(z^\Lambda), \quad G(z) = \sum_{\nu \in \Gamma} z^{-\nu} G_\nu(z^\Lambda) \tag{1}$$

where G_ν and H_ν are the z-transforms of g_ν and h_ν, and $z^\Lambda := [z^{\Lambda_1}, \ldots, z^{\Lambda_n}]^T$ with the column vectors $\Lambda_1, \ldots, \Lambda_n$ of Λ. The *polyphase representation* of the filters h and g are defined as

$$\mathtt{H}(z) := [H_{\nu_0}(z), H_{\nu_1}(z), \ldots, H_{\nu_{q-1}}(z)],$$
$$\mathtt{G}(z) := [G_{\nu_0}(z), G_{\nu_1}(z), \ldots, G_{\nu_{q-1}}(z)]^T.$$

The polyphase representation of analysis and synthesis parts of a FB can be represented by a $p \times q$ matrix $\mathtt{A}(z)$ and a $q \times p$ matrix $\mathtt{S}(z)$, respectively, where p is the number of filters in each bank. In this case, the row vectors of $\mathtt{A}(z)$ represent the polyphase representation of analysis filters, and the column vectors of $\mathtt{S}(z)$ represent the polyphase representation of synthesis filters. Then the perfect reconstruction condition of the FB becomes $\mathtt{S}(z)\mathtt{A}(z) = \mathtt{I}_q$, with $p \geq q$. For non-redundant FBs, the polyphase matrices $\mathtt{A}(z)$ and $\mathtt{S}(z)$ should be $q \times q$ square matrices.

We now briefly review the valid polyphase representation [3] in our context. If we define $v(z) := [1, z^{\nu_1}, \cdots, z^{\nu_{q-1}}]^T$ to be the usual polyphase basis, then from (1), we see that the z-transform of h can be written as

$$H(z) = \mathtt{H}(z^\Lambda) v(z).$$

We recall that $u(z) := \mathtt{M}(z^\Lambda) v(z)$ is called a *valid polyphase basis* if and only if $\mathtt{M}(z)$ is an invertible matrix, i.e. $\mathtt{M}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ (cf. Section II-B). Then the z-transform of the filter can be written using the new basis as

$$H(z) = \mathtt{H}^u(z^\Lambda) u(z),$$

where

$$\mathtt{H}^u(z) := \mathtt{H}(z)[\mathtt{M}(z)]^{-1}$$

is called the *valid (generalized) polyphase representation* of the filter h with respect to the basis $u(z)$.

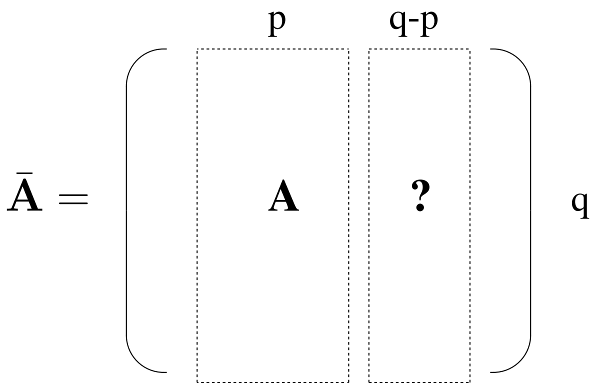

Fig. 1. Unimodular completion of $\mathbf{A}$ to $\bar{\mathbf{A}}$

B. Unimodular vector completion and its use in FB design

Let k be a field and let $k[z^{\pm 1}]$ be the Laurent polynomial ring, consisting of all Laurent polynomials in $z = [z_1, \ldots, z_n]^T$ with coefficients in k. We use $k[z^{\pm 1}]^q$ to denote the set of all column vectors of length q with Laurent polynomial entries in $k[z^{\pm 1}]$, and $\mathrm{GL}_q(k[z^{\pm 1}])$ to denote the set of all invertible $q \times q$ matrices with Laurent polynomial entries in $k[z^{\pm 1}]$. If a matrix with Laurent polynomial entries in $k[z^{\pm 1}]$ is contained in $\mathrm{GL}_q(k[z^{\pm 1}])$, then its inverse exists uniquely and is also contained in $\mathrm{GL}_q(k[z^{\pm 1}])$. In particular, the inverse is also a matrix with Laurent polynomial entries in $k[z^{\pm 1}]$.

A vector $\mathbf{v} = [v_1, \ldots, v_q]$ with Laurent polynomial entries is called *unimodular* if its entries generate 1, i.e. there exist Laurent polynomials $g_1, \ldots, g_q$ such that $v_1 g_1 + \cdots + v_q g_q = 1$. In general, a matrix with Laurent polynomial entries is called a *unimodular matrix* if its maximal minors generate 1.

In 1955, Jean Pierre Serre made a conjecture regarding vector bundles over an affine space [34]. This problem became a daunting task for many mathematicians, and was fully solved only in 1976, 20 years after the question was raised. Serre's conjecture, which is now known as the Quillen-Suslin Theorem ([35], [36]) after the two mathematicians who independently solved this long standing problem, asserts that any unimodular matrix over a polynomial ring can be completed to an invertible square matrix, i.e. a square matrix of nonzero constant determinant. And in 1978, R.G. Swan [4] extended this result to the case of Laurent polynomial rings.

Result 1 (Unimodular Completion, or Quillen-Suslin Theorem for Laurent polynomials): Let $\mathbf{A}$ be a $q \times p$ unimodular matrix, $q \geq p$, with Laurent polynomial entries. Then $\mathbf{A}$ can be completed to a square $q \times q$ unimodular matrix $\bar{\mathbf{A}} \in \mathrm{GL}_q(k[z^{\pm 1}])$ by adding $q - p$ columns to the matrix $\mathbf{A}$. ■

The polyphase representation of a FB consists of the Laurent polynomials in z with real coefficients, which allows many concepts and results in FB design to be stated in terms of these Laurent polynomials. For example, we recall that the two polyphase lowpass filters $\mathtt{H}(z)$ (analysis) and $\mathtt{G}(z)$ (synthesis), or the associated filters h and g, are called *biorthogonal* if $\mathtt{H}(z)\mathtt{G}(z) = 1$, which is equivalent to the row vector $\mathtt{H}(z)$ or the column vector $\mathtt{G}(z)$ being *unimodular*. In such a case, $\mathtt{G}(z)$ (respectively, g) is called a *dual* of $\mathtt{H}(z)$ (respectively, h). Hilbert's Nullstellensatz ([37]) for the Laurent polynomial ring $\mathbb{R}[z^{\pm 1}]$ says that a given row vector $\mathtt{H}(z) = [H_{\nu_0}(z), H_{\nu_1}(z), \ldots, H_{\nu_{q-1}}(z)]$ is unimodular if and only if the Laurent polynomials $H_\nu(z)$, $\nu \in \Gamma$, do not have a nonzero complex common root. Therefore, for a given polyphase analysis lowpass filter $\mathtt{H}(z)$, a dual polyphase synthesis filter $\mathtt{G}(z)$ exists if and only if the components of $\mathtt{H}(z)$ do not have a nonzero complex common root. For a given unimodular polyphase analysis lowpass filter $\mathtt{H}(z)$, Gröbner bases techniques ([38]) can be used to find a particular dual polyphase synthesis lowpass filter, as well as the most general form of dual lowpass filters.

Our method is based on the following special case of the unimodular completion over Laurent polynomial rings:

Result 2 (Corollary of Result 1: Unimodular vector completion): Let $\mathtt{F}(z) \in \mathbb{R}[z^{\pm 1}]^q$ be a unimodular column vector of length q. Then there exists an invertible $q \times q$ matrix $\mathtt{K}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ such that $\mathtt{K}(z)\mathtt{F}(z) = [1, 0, \ldots, 0]^T$. ■

While the original proofs of Quillen-Suslin Theorem were nonconstructive, algorithmic proofs were studied in [39]–[41]. By using these algorithms, given a unimodular polynomial vector $\mathtt{F}(z)$, one can compute a companion unimodular polynomial matrix $\mathtt{K}(z)$ in Result 2. This algorithm was extended to unimodular Laurent polynomial matrices in [42], which was implemented as a part of the Maple package *QuillenSuslin* by Anna Fabiańska (see http://wwwb.math.rwth-aachen.de/QuillenSuslin/).

There have been many studies on the design of multi-D FBs using unimodular completion (cf. Section I), but there was little success in developing a simple construction method for wavelet FBs, not just FBs. In other words, how one can make sure the resulting FB to have a certain number of vanishing moments, without much work, has been a remaining challenge for the most part. Our approach in this paper provides an answer to this question.

It is well known that (see, for example, [17]) the number of vanishing moments of the non-redundant wavelet FB is at least s if the accuracy numbers of its lowpass filters are at least s. We recall that for a given lowpass filter f, the number of zeros of the Fourier transform of f at $\omega \in \Gamma^* \backslash \{0\}$ is referred to as the *accuracy number* [31]. This number determines the maximum degree of polynomials that can be reproduced by the filter f and it is closely related with the Strang-Fix order in the wavelet theory [43]. When a wavelet FB gives rise to a wavelet system in $L^2(\mathbb{R}^n)$, the number of vanishing moments of the wavelet system is completely determined by the (discrete) vanishing moments of the wavelet FB. Therefore, for constructing multi-D wavelet bases with a certain number of vanishing moments, we can start from the two biorthogonal lowpass filters with prescribed accuracy numbers. Unfortunately, this too is not easy in general and requires great care in the construction process. Our result (Corollary 2) presented in the next section provides a solution to this problem.

III. Construction of Multi-D Wavelet FBs using Quillen-Suslin Theorem

In this section, we present a new method for constructing multi-D wavelets using the Quillen-Suslin Theorem for Laurent polynomials. From this method, algorithms for constructing a non-redundant multi-D wavelet FB just from a single lowpass filter can be obtained. The motivation and the main idea of our method is presented in Section III-A, the main results are shown in Section III-B, and the algorithms are shown in Section III-C.

A. Motivation

Many of the existing construction methods for multi-D wavelet systems ([17], [18], [21], [26]–[28]) assume that at least one of the lowpass filters is interpolatory. We recall that a lowpass filter f is *interpolatory* if

$$f(0)=\frac{1}{\sqrt{q}} \quad \text{and} \quad f(\Lambda m)=0, \forall m \in \mathbb{Z}^n \backslash \{0\}.$$

Equivalently, the polyphase lowpass filter $\mathtt{F}(z)$ is interpolatory if its first component satisfies

$$F_{\nu_0}(z)=\frac{1}{\sqrt{q}}.$$

It is easy to see that every polyphase interpolatory lowpass filter $\mathtt{F}(z)$ is unimodular, since the dual vector can be chosen so that its first component is $\sqrt{q}$ and the rest are all zero.

The Laplacian pyramid (LP) representation ([44]) has been used in many image processing applications [45]–[47]. In the LP algorithms, if the interpolatory lowpass filter h is used for analysis and the "lazy" interpolatory ([48]) lowpass filter g is used for synthesis as its dual, then we have ([49])

$$\mathtt{H}(z)=[\frac{1}{\sqrt{q}}, H_{\nu_1}(z), \cdots, H_{\nu_{q-1}}(z)], \quad \mathtt{G}(z)=[\sqrt{q}, 0, \cdots, 0]^T$$

and

$$\begin{bmatrix} \mathtt{G}(z) & \mathtt{I}_q \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z) \end{bmatrix} = \mathtt{I}_q.$$

Although the above matrices can be considered as a polyphase representation of a redundant FB, it is clear that this FB is not a wavelet FB as the synthesis filters associated with the column vectors of the polyphase matrix $\mathtt{I}_q$ do not have any vanishing moment. A new method called the interpolatory effortless critical representation of LP is proposed in order to transform these LP-based *redundant non-wavelet* FBs to *non-redundant wavelet* FBs in a remarkably simple way [26]. This new method provides a way to construct non-redundant wavelet FBs for any dimension and any dilation. A critical assumption for this method is that $\mathtt{H}(z)$ has to be essentially interpolatory (see (23) in [26] for a precise statement of the assumption).

A closer look at the interpolatory lowpass filter reveals that not only its polyphase representation $\mathtt{H}(z)$ is unimodular, but also it has a dual with a unit in at least one of its components. We recall that an element in a ring is called a *unit* if its multiplicative inverse lies in the ring. Scrutinizing the techniques used in [26] shows that many arguments used there rely on this "nice" property of analysis interpolatory lowpass filters. Therefore, at first sight, it may appear to be difficult to directly apply them to more general analysis lowpass filters.

On the other hand, we notice that many techniques used in [26] work regardless of where the Laurent polynomial matrices are coming from. The key idea of our new construction method is to decompose the z-transform of filters using a special type of the valid polyphase representations obtained by unimodular vector completion over Laurent polynomial rings. In some sense, this can be understood as a change of basis, from the usual polyphase basis to the valid polyphase basis, in the Laurent polynomial ring. In the next subsection, we show exactly how this new representation is obtained.

B. Main results

Our new construction method relies on Result 2. In fact, the following slightly modified version of Result 2 is sufficient for the arguments in the proof and it gives more flexibility in the construction process.

Result 3 (A slightly modified version of Result 2): Let $\mathtt{F}(z) \in \mathbb{R}[z^{\pm 1}]^q$ be a unimodular column vector of length q. Then there exists an invertible $q \times q$ matrix $\mathtt{T}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ such that $\mathtt{T}(z)\mathtt{F}(z)$ is a unimodular column vector that has a unit in at least one of its components. ■

Our main theorem is placed below. It provides the theory and the algorithm to construct a non-redundant wavelet FB from a lowpass filter whose polyphase representation is unimodular. It uses Result 3 and part of the arguments used to prove some results (Theorem 1 and 2) in [26]. It is also a variant of a result[1] (Theorem 1) in [50].

Theorem 1: Let h be a lowpass filter with positive accuracy. If its polyphase representation $\mathtt{H}(z)$ as a row vector is unimodular, then there exists a non-redundant wavelet FB whose analysis lowpass filter is h. ■

Proof 1: Since $\mathtt{H}(z)=[H_{\nu_0}(z), \ldots, H_{\nu_{q-1}}(z)]$ is unimodular, there exists $\mathtt{F}(z)=[F_{\nu_0}(z), \ldots, F_{\nu_{q-1}}(z)]^T$ such that

$$\mathtt{H}(z)\mathtt{F}(z)=H_{\nu_0}(z)F_{\nu_0}(z)+\ldots+H_{\nu_{q-1}}(z)F_{\nu_{q-1}}(z)=1.$$

Thus $\mathtt{F}(z)$ is also unimodular. By Result 3, there exists an invertible $q \times q$ matrix $\mathtt{T}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ such that $\mathtt{T}(z)\mathtt{F}(z)$ is a unimodular vector with a unit in at least one of its components. Without loss of generality, we assume the first component of $\mathtt{T}(z)\mathtt{F}(z)$ is a unit.

Let g be another lowpass filter with positive accuracy that can possibly be different from h, and let $\mathtt{G}(z) := [G_{\nu_0}(z), G_{\nu_1}(z), \ldots, G_{\nu_{q-1}}(z)]^T$ be its synthesis polyphase representation.

From the discussion at the end of Section II-A, we see that the z-transform of h, f and g can be written as

$$\begin{aligned} H(z) &= \mathtt{H}(z^\Lambda)v(z), \\ F(z) &= v(z)^*\mathtt{F}(z^\Lambda), \\ G(z) &= v(z)^*\mathtt{G}(z^\Lambda), \end{aligned}$$

[1] While the statement of Theorem 1 in [50] is correct, the proof presented there turns out to contain an error.

where $v(z) = [1, z^{\nu_1}, \cdots, z^{\nu_{q-1}}]^T$ is the usual polyphase basis as before, and $v(z)^* := v(z^{-1})^T$ is the conjugate transpose of $v(z)$.

We take the approach in [3] but extend it slightly by allowing two different valid polyphase bases for analysis and synthesis filters. More precisely, using the above invertible matrix $\mathtt{T}(z)$, we define a new pair of valid polyphase bases $u(z) := \mathtt{T}(z^\Lambda)v(z)$ and $w(z) := [\mathtt{T}(z^\Lambda)^*]^{-1}v(z)$, and use them instead of the usual basis $v(z)$ to represent the z-transform of the analysis and the synthesis filters, respectively. For example,

$$\begin{aligned} H(z) &= \mathtt{H}^u(z^\Lambda)u(z), \\ F(z) &= w(z)^*\mathtt{F}^w(z^\Lambda), \\ G(z) &= w(z)^*\mathtt{G}^w(z^\Lambda), \end{aligned}$$

where

$$\begin{aligned} \mathtt{H}^u(z) &:= \mathtt{H}(z)[\mathtt{T}(z)]^{-1}, \\ \mathtt{F}^w(z) &:= \mathtt{T}(z)\mathtt{F}(z), \\ \mathtt{G}^w(z) &:= \mathtt{T}(z)\mathtt{G}(z) \end{aligned}$$

are the valid polyphase representation of h, f and g with respect to the new valid polyphase basis pair $(u(z), w(z))$.

Then from the fact that $\mathtt{F}^w(z)$ is a particular dual to $\mathtt{H}^u(z)$, i.e. $\mathtt{H}^u(z)\mathtt{F}^w(z) = \mathtt{H}(z)\mathtt{F}(z) = 1$, we see that any column vector of the form $\mathtt{G}^w(z) + \mathtt{F}^w(z)(1 - \mathtt{H}^u(z)\mathtt{G}^w(z))$ is also dual to $\mathtt{H}^u(z)$. In fact, it is easy to see that the matrix identity

$$\begin{bmatrix} \mathtt{D}^w(z) & \mathtt{I}_q - \mathtt{F}^w(z)\mathtt{H}^u(z) \end{bmatrix} \begin{bmatrix} \mathtt{H}^u(z) \\ \mathtt{I}_q - \mathtt{G}^w(z)\mathtt{H}^u(z) \end{bmatrix} = \mathtt{I}_q \tag{2}$$

always holds true, where $\mathtt{D}^w(z) := \mathtt{G}^w(z) + \mathtt{F}^w(z)(1 - \mathtt{H}^u(z)\mathtt{G}^w(z))$.

Since $F^w_{\nu_0}(z)$, the first component of $\mathtt{F}^w(z)$, is assumed to be a unit, if we define

$$\mathtt{R}(z) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c(z)F^w_{\nu_0}(z) & & & \\ 0 & c(z)F^w_{\nu_1}(z) & 1 & & \\ 0 & \vdots & & \ddots & \\ 0 & c(z)F^w_{\nu_{q-1}}(z) & & & 1 \end{bmatrix}$$

to be the $(q+1) \times (q+1)$ reduction matrix with any unit $c(z)$ in the Laurent polynomial ring $\mathbb{R}[z^{\pm 1}]$, then the second column of

$$\begin{bmatrix} \mathtt{D}^w(z) & \mathtt{I}_q - \mathtt{F}^w(z)\mathtt{H}^u(z) \end{bmatrix} \mathtt{R}(z)$$

becomes a zero column vector. Since the reduction matrix $\mathtt{R}(z)$ is invertible, i.e. $\mathtt{R}(z) \in \mathrm{GL}_{q+1}(\mathbb{R}[z^{\pm 1}])$, by inserting $\mathtt{R}(z)[\mathtt{R}(z)]^{-1}$ between the two matrices on the left-hand side of (2), we get

$$[\mathtt{D}^w(z), \mathtt{I}_q{-}\mathtt{F}^w(z)\mathtt{H}^u(z)]\mathtt{R}(z)[\mathtt{R}(z)]^{-1} \begin{bmatrix} \mathtt{H}^u(z) \\ \mathtt{I}_q - \mathtt{G}^w(z)\mathtt{H}^u(z) \end{bmatrix} {=} \mathtt{I}_q \tag{3}$$

By defining $\mathtt{S}(z)$ to be the $q \times q$ matrix obtained by deleting the second column of the product of the first two matrices on the left-hand side, and $\mathtt{A}(z)$ to be the $q \times q$ matrix obtained by deleting the second row of the product of the last two matrices on the left-hand side, we get a non-redundant FB with $\mathtt{S}(z)\mathtt{A}(z) = \mathtt{I}_q$.

Since the first row of $[\mathtt{R}(z)]^{-1}$ is $[1, 0, \cdots, 0]$, the first row of the analysis polyphase matrix $\mathtt{A}(z)$ is $\mathtt{H}^u(z)$, which in turn implies that the analysis lowpass filter is h in the above non-redundant FB. In order to finish the proof, we need to show that the non-redundant FB obtained above is a wavelet FB. It suffices to show that both the analysis lowpass filter h and the synthesis lowpass filter, say d, have positive accuracy (cf. Section II-B). Since h has positive accuracy by the assumption, we only need to show that d has positive accuracy. Since its polyphase representation satisfies

$$\begin{aligned} \mathtt{D}(z) &= [\mathtt{T}(z)]^{-1}\mathtt{D}^w(z) \\ &= [\mathtt{T}(z)]^{-1}(\mathtt{G}^w(z) + \mathtt{F}^w(z)(1 - \mathtt{H}^u(z)\mathtt{G}^w(z))) \\ &= \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z)), \end{aligned}$$

and since both h and g are assumed to have positive accuracy, we have $\mathtt{D}(1) = \mathtt{G}(1) + \mathtt{F}(1)(1 - \mathtt{H}(1)\mathtt{G}(1)) = \frac{1}{\sqrt{q}}[1, \ldots, 1]^T + \mathtt{F}(1)(1 - \frac{1}{\sqrt{q}}[1, \ldots, 1]\frac{1}{\sqrt{q}}[1, \ldots, 1]^T) = \frac{1}{\sqrt{q}}[1, \ldots, 1]^T$, from which we can conclude that d also has positive accuracy (cf. Result 2 in [26]). ■

Remark 1: Although we stated Theorem 1 for the case when the lowpass filter is used for the analysis, a similar statement can be made for the synthesis lowpass filter. ■

Remark 2: It is easy to see that the converse of the statement of Theorem 1 is also true. ■

Remark 3: All the filters in the resulting non-redundant wavelet FB of Theorem 1 are FIR filters. This is because the matrices $[\mathtt{R}(z)]^{-1}$ and $[\mathtt{T}(z)]^{-1}$ appearing in the proof have Laurent polynomial entries in $\mathbb{R}[z^{\pm 1}]$ (cf. Section II-B). ■

Although the construction method developed in the above theorem works for any dimension and for any dilation, it is especially useful for the wavelet construction in multi-D setting as this is where the problem gets more challenging. We now present 2-D examples to illustrate our findings. For simplicity, in all of our examples, we consider the dyadic dilation and choose $\Gamma = \{(0,0), (1,0), (0,1), (1,1)\}$.

Example 1 (2-D wavelet FB generated from an interpolatory lowpass filter). Let h be the lowpass filter associated with the bivariate piecewise-linear box spline $B_{1,1,1}$ based on the three directions $(1,0), (0,1)$, and $(1,1)$ (see [51] for the definition of box splines and their properties), i.e.

$$h: \quad \begin{array}{cccc} & \frac{1}{4} & \frac{1}{4} & \\ \frac{1}{4} & \boxed{\frac{1}{2}} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & & \end{array}$$

Here and below, the number in the box represents the coefficient of the filter at the origin. Since h is interpolatory and its polyphase representation is

$$\mathtt{H}(z) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4}z_1^{-1} + \frac{1}{4} & \frac{1}{4}z_2^{-1} + \frac{1}{4} & \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4} \end{bmatrix},$$

we can choose $\mathtt{F}(z) = [\ 2 \quad 0 \quad 0 \quad 0\ ]^T$. If we take $g = h$ and $\mathtt{T}(z) = \mathtt{I}_4$, then the matrix identity (2) becomes

$$\begin{bmatrix} \mathtt{D}(z) & \mathtt{I}_4 - \mathtt{F}(z)\mathtt{H}(z) \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{I}_4 - \mathtt{H}^*(z)\mathtt{H}(z) \end{bmatrix} = \mathtt{I}_4 \tag{4}$$

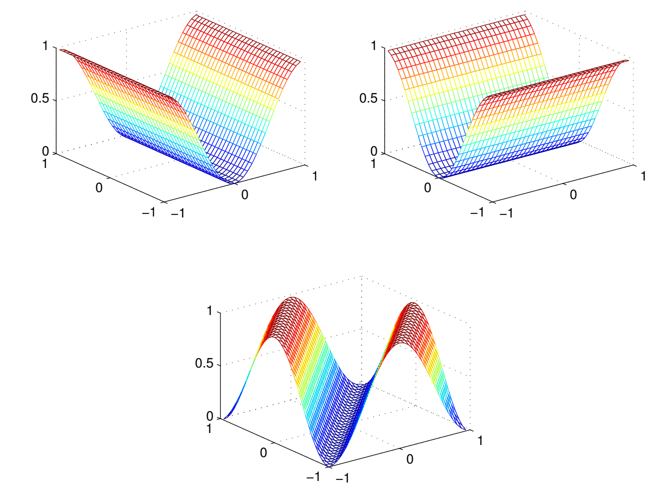

Fig. 2. The magnitude of the frequency responses of the filters k_1, k_2, k_3 in Example 1.

where $\mathtt{D}(z) := \mathtt{H}^*(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{H}^*(z))$ and $\mathtt{H}^*(z) = \mathtt{H}(z^{-1})^T$ is the conjugate transpose of $\mathtt{H}(z)$. Hence, from the arguments in the proof of Theorem 1, we obtain a non-redundant wavelet FB. Let $\mathtt{A}(z)$ be its analysis polyphase matrix. Then the first row of $\mathtt{A}(z)$ is $\mathtt{H}(z)$, and the second through the fourth rows of $\mathtt{A}(z)$ are the transpose of the following column vectors

$$\begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_1 \\ -\frac{1}{16}z_1^{-1} + \frac{7}{8} - \frac{1}{16}z_1 \\ -\frac{1}{16}z_2^{-1} - \frac{1}{16}z_2^{-1}z_1 - \frac{1}{16} - \frac{1}{16}z_1 \\ -\frac{1}{16}z_1^{-1}z_2^{-1} - \frac{1}{16}z_2^{-1} - \frac{1}{16} - \frac{1}{16}z_1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_2 \\ -\frac{1}{16}z_1^{-1} - \frac{1}{16}z_1^{-1}z_2 - \frac{1}{16} - \frac{1}{16}z_2 \\ -\frac{1}{16}z_2^{-1} + \frac{7}{8} - \frac{1}{16}z_2 \\ -\frac{1}{16}z_1^{-1}z_2^{-1} - \frac{1}{16}z_1^{-1} - \frac{1}{16} - \frac{1}{16}z_2 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_1z_2 \\ -\frac{1}{16}z_1^{-1} - \frac{1}{16} - \frac{1}{16}z_2 - \frac{1}{16}z_1z_2 \\ -\frac{1}{16}z_2^{-1} - \frac{1}{16} - \frac{1}{16}z_1 - \frac{1}{16}z_1z_2 \\ -\frac{1}{16}z_1^{-1}z_2^{-1} + \frac{7}{8} - \frac{1}{16}z_1z_2 \end{bmatrix},$$

respectively. Its synthesis polyphase matrix $\mathtt{S}(z)$ is given as

$$\begin{bmatrix} \alpha(z_1,z_2) & -\frac{1}{2}z_1^{-1} - \frac{1}{2} & -\frac{1}{2}z_2^{-1} - \frac{1}{2} & -\frac{1}{2}z_1^{-1}z_2^{-1} - \frac{1}{2} \\ \frac{1}{4} + \frac{1}{4}z_1 & 1 & 0 & 0 \\ \frac{1}{4} + \frac{1}{4}z_2 & 0 & 1 & 0 \\ \frac{1}{4} + \frac{1}{4}z_1z_2 & 0 & 0 & 1 \end{bmatrix}$$

where $\alpha(z_1,z_2) = \frac{1}{2} + 2(\frac{3}{8} - \frac{1}{16}(z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1} + z_1 + z_2 + z_1z_2))$. In particular, the three synthesis highpass filters, say k_1, k_2, and k_3, are directional and aligned along the directions determined by the nonzero cosets $(1,0)$, $(0,1)$, $(1,1)$, i.e.,

$$k_1: \quad \begin{array}{ccc} \boxed{-\frac{1}{2}} & 1 & -\frac{1}{2} \end{array} \qquad\qquad k_2: \quad \begin{array}{c} -\frac{1}{2} \\ 1 \\ \boxed{-\frac{1}{2}} \end{array}$$

$$k_3: \quad \begin{array}{ccc} & & -\frac{1}{2} \\ & 1 & \\ \boxed{-\frac{1}{2}} & & \end{array}$$

The magnitude of the frequency responses of these highpass filters are depicted in Figure 2. ■

Example 1 provides a simple way to construct a 2-D non-redundant wavelet FB from an interpolatory lowpass filter h, and the resulting synthesis highpass filters are directional and very sparse. Since h is interpolatory, in principle, other existing methods (e.g. methods in [26], [28]) that work under the interpolatory condition may be used to give a similar result. In the next example, we show how our method can be used to construct a non-redundant wavelet FB from a non-interpolatory lowpass filter h.

Example 2 (2-D wavelet FB generated from a non-interpolatory lowpass filter). Let h be the lowpass filter associated with the bivariate box spline $B_{1,1,2}$ based on the four directions $(1,0), (0,1)$, $(1,1)$ and $(1,1)$, i.e.

$$h: \quad \begin{array}{cccc} & & \frac{1}{8} & \frac{1}{8} \\ & \frac{1}{4} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \boxed{\frac{3}{8}} & \frac{1}{4} & \\ \frac{1}{8} & \frac{1}{8} & & \end{array}$$

Then the filter h is no longer interpolatory and its polyphase representation $\mathtt{H}(z)$ is $[\frac{3}{8} + \frac{1}{8}z_1^{-1}z_2^{-1}, \frac{1}{8} + \frac{1}{4}z_1^{-1} + \frac{1}{8}z_1^{-1}z_2^{-1}, \frac{1}{8} + \frac{1}{4}z_2^{-1} + \frac{1}{8}z_1^{-1}z_2^{-1}, \frac{1}{8} + \frac{3}{8}z_1^{-1}z_2^{-1}]$. We choose $\mathtt{F}(z) = [\begin{array}{cccc} 3 & 0 & 0 & -1 \end{array}]^T$ as a dual of $\mathtt{H}(z)$. As we did in Example 1, we take $g = h$ and $\mathtt{T}(z) = \mathtt{I}_4$. Then we obtain the same identity as in (4) of Example 1 for our new $\mathtt{F}(z)$ in this example. By using the arguments in the proof of Theorem 1 again, we obtain a non-redundant wavelet FB. Let $\mathtt{A}(z)$ be its analysis polyphase matrix. Then the first row of $\mathtt{A}(z)$ is $\mathtt{H}(z)$, whereas the second through the fourth rows of $\mathtt{A}(z)$ are the transpose of the following column vectors

$$\begin{bmatrix} -\frac{1}{16} - \frac{3}{32}z_1 - \frac{1}{32}z_2^{-1} - \frac{3}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} \\ \frac{29}{32} - \frac{1}{32}z_1 - \frac{1}{32}z_1^{-1} - \frac{1}{32}z_2 - \frac{1}{32}z_2^{-1} - \frac{1}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} \\ -\frac{1}{32} - \frac{1}{16}z_1 - \frac{1}{16}z_2^{-1} - \frac{1}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} - \frac{1}{16}z_1z_2^{-1} \\ -\frac{1}{16} - \frac{1}{32}z_1 - \frac{3}{32}z_2^{-1} - \frac{1}{64}z_1z_2 - \frac{3}{64}z_1^{-1}z_2^{-1} \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{16} - \frac{1}{32}z_1^{-1} - \frac{3}{32}z_2 - \frac{3}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} \\ -\frac{1}{32} - \frac{1}{16}z_1^{-1} - \frac{1}{16}z_2 - \frac{1}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} - \frac{1}{16}z_1^{-1}z_2 \\ \frac{29}{32} - \frac{1}{32}z_1 - \frac{1}{32}z_1^{-1} - \frac{1}{32}z_2 - \frac{1}{32}z_2^{-1} - \frac{1}{64}z_1z_2 - \frac{1}{64}z_1^{-1}z_2^{-1} \\ -\frac{1}{16} - \frac{3}{32}z_1^{-1} - \frac{1}{32}z_2 - \frac{1}{64}z_1z_2 - \frac{3}{64}z_1^{-1}z_2^{-1} \end{bmatrix},$$

$$\begin{bmatrix} \frac{3}{16} - \frac{5}{32}z_1z_2 - \frac{1}{32}z_1^{-1}z_2^{-1} \\ -\frac{1}{12} - \frac{1}{16}z_1^{-1} - \frac{5}{48}z_2 - \frac{5}{96}z_1z_2 - \frac{1}{32}z_1^{-1}z_2^{-1} \\ -\frac{1}{12} - \frac{5}{48}z_1 - \frac{1}{16}z_2^{-1} - \frac{5}{96}z_1z_2 - \frac{1}{32}z_1^{-1}z_2^{-1} \\ \frac{13}{16} - \frac{5}{96}z_1z_2 - \frac{3}{32}z_1^{-1}z_2^{-1} \end{bmatrix},$$

respectively. The first column of its synthesis polyphase matrix $\mathtt{S}(z)$ is

$$\begin{bmatrix} \frac{3}{8} + \frac{1}{8}z_1z_2 + 3(\frac{1}{2} - \frac{1}{16}(z_1^{-1} + z_2^{-1} + 2z_1^{-1}z_2^{-1} + z_1 + z_2 + 2z_1z_2)) \\ \frac{1}{8} + \frac{1}{4}z_1 + \frac{1}{8}z_1z_2 \\ \frac{1}{8} + \frac{1}{4}z_2 + \frac{1}{8}z_1z_2 \\ \frac{1}{8} + \frac{3}{8}z_1z_2 - (\frac{1}{2} - \frac{1}{16}(z_1^{-1} + z_2^{-1} + 2z_1^{-1}z_2^{-1} + z_1 + z_2 + 2z_1z_2)) \end{bmatrix},$$

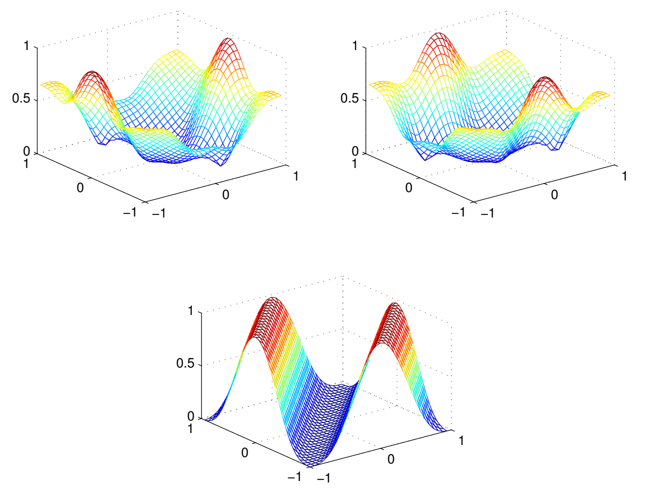

Fig. 3. The magnitude of the frequency responses of the filters k_1, k_2, k_3 in Example 2.

whereas the second through the fourth columns of $\mathtt{S}(z)$ are the following column vectors

$$\begin{bmatrix} -\frac{3}{8} - \frac{3}{4}z_1^{-1} - \frac{3}{8}z_1^{-1}z_2^{-1} \\ 1 \\ 0 \\ \frac{1}{8} + \frac{1}{4}z_1^{-1} + \frac{1}{8}z_1^{-1}z_2^{-1} \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{8} - \frac{3}{4}z_2^{-1} - \frac{3}{8}z_1^{-1}z_2^{-1} \\ 0 \\ 1 \\ \frac{1}{8} + \frac{1}{4}z_2^{-1} + \frac{1}{8}z_1^{-1}z_2^{-1} \end{bmatrix},$$

$$\begin{bmatrix} -\frac{3}{8} - \frac{9}{8}z_1^{-1}z_2^{-1} \\ 0 \\ 0 \\ \frac{9}{8} + \frac{3}{8}z_1^{-1}z_2^{-1} \end{bmatrix},$$

respectively. The three synthesis highpass filters, k_1, k_2, and k_3 are given as

$$k_1: \quad \begin{array}{cccc} & & & \frac{1}{8} \\ & & -\frac{3}{8} & \\ & \frac{1}{8} & & \frac{1}{4} \\ \boxed{-\frac{3}{8}} & 1 & -\frac{3}{4} & \end{array} \qquad k_2: \quad \begin{array}{cccc} & \frac{1}{4} & & \frac{1}{8} \\ -\frac{3}{4} & & -\frac{3}{8} & \\ 1 & \frac{1}{8} & & \\ \boxed{-\frac{3}{8}} & & & \end{array}$$

$$k_3: \quad \begin{array}{cccc} & & & \frac{3}{8} \\ & & -\frac{9}{8} & \\ & \frac{9}{8} & & \\ \boxed{-\frac{3}{8}} & & & \end{array}$$

and the magnitude of their frequency responses are drawn in Figure 3. ■

Below we list two corollaries of Theorem 1, whose proofs are placed in Appendix. The first corollary says that the accuracy number of the synthesis lowpass filter of the non-redundant wavelet FB in Theorem 1 can be stated in terms of the accuracy number and the flatness number of the other filters involved in the construction. Here, the *flatness number* of a filter f is defined to be the number of zeros of $\sqrt{q} - F(e^{i\omega})$ at $\omega = 0$. Notice that f is a lowpass filter if and only if its flatness number is positive.

Corollary 1: Let h be a lowpass filter with flatness β_h. Suppose that h has a dual lowpass filter. Let f be a dual lowpass filter of h with accuracy α_f, and let g be a lowpass filter with accuracy α_g and flatness β_g. Suppose that the accuracy number α_g is positive. Then there exists a dual lowpass filter d of h such that the filter d is determined entirely from f, g, and h, and that the accuracy of the filter d is at least $\min\{\alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$. ■

In the above corollary, the dual filter d has positive accuracy since $\min\{\alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$ is clearly positive, which in turn is implied by the positivity of α_g, β_g, and β_h. However, $\min\{\alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$ may be lagging behind α_h, the accuracy number of the lowpass filter h. In such a case, one may want to find a dual whose accuracy number is at least α_h. The next corollary says that such a dual can always be found.

Corollary 2: Let h be a lowpass filter with positive accuracy α_h. Suppose that h has a dual lowpass filter f. Then there exists a dual lowpass filter d of h such that the filter d is determined entirely from f and h, and that the accuracy of the filter d is at least α_h. ■

As we observed in the previous subsection, a new method developed in [26] provides a motivation for our construction method presented in this paper. Indeed, the fact that it is a special case of our general construction can be shown as follows. We recall the polyphase representation of an interpolatory analysis lowpass filter is given as $\mathtt{H}(z) = [\frac{1}{\sqrt{q}}, H_{\nu_1}(z), \cdots, H_{\nu_{q-1}}(z)]$. Thus we can set $\mathtt{F}(z) = [\sqrt{q}, 0, \cdots, 0]^T$ and $\mathtt{T}(z) = \mathtt{I}_q$ (cf. Example 1). Therefore, in this case, no change of basis is needed and the usual polyphase representation is sufficient. The first matrix on the left-hand side of identity (2) in this case becomes

$$\begin{bmatrix} \mathtt{D}(z) & \begin{matrix} 0 & -\sqrt{q}H_{\nu_1}(z) & \cdots & -\sqrt{q}H_{\nu_{q-1}}(z) \\ 0 & & & \\ \vdots & & \mathtt{I}_{q-1} & \\ 0 & & & \end{matrix} \end{bmatrix},$$

where $\mathtt{D}(z) := \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$. The second matrix on the left-hand side of identity (2) becomes

$$\begin{bmatrix} \mathtt{H}(z) \\ \mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z) \end{bmatrix}.$$

By deleting the second column of the first matrix and the second row of the second matrix, we obtain the non-redundant wavelet FB in [26]. Hence our result here can be considered as a generalization of the method in the aforementioned paper.

C. Algorithms for constructing multi-D wavelet FBs from a single lowpass filter

Our methodology in the previous subsection is very general. In particular, the filters f, g, and h in Corollary 1 or 2 do not, in general, uniquely determine the highpass filters of the associated wavelet FB, which may not be desirable for some applications. The following corollary provides a way to obtain unique highpass filters given f, g, and h by choosing the matrix $\mathtt{T}(z)$ in the proof of Theorem 1 to be a special form. Its proof is placed in Appendix.

Corollary 3: Let h be a lowpass filter with accuracy α_h and flatness β_h. Suppose that h has a dual lowpass filter. Let f be a dual lowpass filter of h with accuracy α_f, and let g be a lowpass filter with accuracy α_g and flatness β_g. Suppose that the accuracy numbers α_h and α_g are positive. Let $\mathtt{K}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ be an invertible $q \times q$ matrix such that $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, \cdots, 0]^T$ where $\mathtt{H}(z)$ (as a row vector) is the polyphase representation of h. Let d be the filter whose polyphase representation is $\mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$ where $\mathtt{G}(z)$ and $\mathtt{F}(z)$ are the polyphase representation (as a column vector) of g and f. Let $k_1, \ldots, k_{q-1}$ and $j_1, \ldots, j_{q-1}$ be the filters whose polyphase representations are the 2nd through the qth column of $\mathtt{K}(z)^T$ and the 2nd through the qth row of $[\mathtt{K}(z)^T]^{-1}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$, respectively. Then $\{h, j_1, \ldots, j_{q-1}\}$, $\{d, k_1, \ldots, k_{q-1}\}$ form a wavelet FB with at least $\min\{\alpha_h, \alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$ vanishing moments. ■

The above corollary provides an algorithm to construct a non-redundant wavelet FB just from a single lowpass filter h, provided that h has positive accuracy and its polyphase representation $\mathtt{H}(z)$ is unimodular. We note that this positive accuracy condition on h and the unimodularity condition on $\mathtt{H}(z)$ are necessary conditions for any lowpass filter to be used for wavelet FBs. In this sense, one can say that our algorithms below work under the minimum assumptions on the lowpass filter h.

Algorithm 1: An algorithm for constructing a non-redundant wavelet FB from a lowpass filter.

Input: *h: a lowpass filter with positive accuracy and with unimodular polyphase representation.*

Output: *d: a dual lowpass filter of h with positive accuracy.*
$j_1, \ldots, j_{q-1}, k_1, \ldots, k_{q-1}$: highpass filters that form a wavelet FB, together with h and d.

Step 1: *Choose a lowpass filter g with positive accuracy.*

Step 2: *Find a lowpass filter f that is dual to h.*

Step 3: *Find an invertible $q \times q$ matrix $\mathtt{K}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ such that $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, \cdots, 0]^T$ where $\mathtt{H}(z)$ (as a row vector) is the polyphase representation of h.*

Step 4: *Set d to be the filter whose polyphase representation is $\mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$ where $\mathtt{G}(z)$ and $\mathtt{F}(z)$ are the polyphase representation (as a column vector) of g and f.*

Step 5: *Set $k_1, \ldots, k_{q-1}$ to be the filters whose polyphase representations are the 2nd through the qth column vectors of $\mathtt{K}(z)^T$.*

Step 6: *Set $j_1, \ldots, j_{q-1}$ to be the filters whose polyphase representations are the 2nd through the qth row vectors of the matrix $[\mathtt{K}(z)^T]^{-1}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$.* ■

The above algorithm starts from a given lowpass filter h to build a wavelet FB, whose analysis lowpass filter is h. Since the matrices $\mathtt{K}(z)$, $\mathtt{K}(z)^T$ and $[\mathtt{K}(z)^T]^{-1}$ are all contained in $\mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$, the resulting wavelet filters are all FIR filters. The filter g in Step 1 is an arbitrary lowpass filter with positive accuracy. One possible choice is to take $g := h$ as we did in our examples in the previous subsection. The existence of f in Step 2 and $\mathtt{K}(z)$ in Step 3 is due to the facts that h has positive accuracy and $\mathtt{H}(z)$ is unimodular. In fact, one can always choose f to be the filter whose polyphase representation is the first column vector of $\mathtt{K}(z)^T$ once $\mathtt{K}(z)$ is determined. Although algorithms for finding f and $\mathtt{K}(z)$ are implemented in many mathematical softwares such as Maple, Singular and CoCoA, the *QuillenSuslin* package in Maple (cf. Section II-B) is the only implementation that we know to give a square matrix $\mathtt{K}(z)$ for any unimodular $\mathtt{H}(z)$, which is important for our algorithms to work through smoothly. Given h, once specific f, g and $\mathtt{K}(z)$ are chosen, the wavelet FB having h as its analysis lowpass filter is uniquely determined.

From Corollary 3, we see that the vanishing moments of the FBs constructed following Algorithm 1 are at least $\min\{\alpha_h, \alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$. Although this number is clearly positive, which is enough for the FB to be a wavelet FB, it can be lagging behind α_h. By combining Corollary 3 (or Algorithm 1) with the idea used in Corollary 2, one can obtain the following algorithm that provides wavelet FBs whose vanishing moments are at least α_h.

Algorithm 2: An algorithm for constructing a non-redundant wavelet FB from a lowpass filter so that its vanishing moments are at least as many as the accuracy number of the lowpass filter.

Input: *h: a lowpass filter with positive accuracy α_h and with unimodular polyphase representation.*

Output: *Ite: the number of iterations performed.*
d: a dual lowpass filter of h with positive accuracy.
$j_1, \ldots, j_{q-1}, k_1, \ldots, k_{q-1}$: highpass filters that form, together with h and d, a wavelet FB with at least α_h vanishing moments.

Step 1: *Set $Ite := 1$ and $g := h$.*

Step 2: *Find a lowpass filter f that is dual to h.*

Step 3: *Find an invertible $q \times q$ matrix $\mathtt{K}(z) \in \mathrm{GL}_q(\mathbb{R}[z^{\pm 1}])$ such that $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, \cdots, 0]^T$ where $\mathtt{H}(z)$ (as a row vector) is the polyphase representation of h.*

Step 4: *Set d to be the filter whose polyphase representation is $\mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$ where $\mathtt{G}(z)$ and $\mathtt{F}(z)$ are the polyphase representation (as a column vector) of g and f.*

Step 5: *If $\alpha_f + (Ite)\beta_h < \alpha_h$, set $Ite := Ite + 1$ and repeat* **Step 4** *with $f := d$. Otherwise, go to* **Step 6**.

Step 6: *Set $k_1, \ldots, k_{q-1}$ to be the filters whose polyphase representations are the 2nd through the qth column vectors of $\mathtt{K}(z)^T$.*

Step 7: *Set $j_1, \ldots, j_{q-1}$ to be the filters whose polyphase representations are the 2nd through the qth row vectors of the matrix $[\mathtt{K}(z)^T]^{-1}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$.* ■

IV. Summary and Outlook

In this paper we presented a new algebraic approach for constructing wavelet FBs using Quillen-Suslin Theorem for Laurent polynomials. Our method is motivated by some existing techniques that were used mostly only for interpolatory filters (cf. Section III-A). Quillen-Suslin Theorem for Laurent polynomials is used to transform the filters in polyphase representation to a special form of valid polyphase representations, for which the existing matrix analysis tools can be readily applied (cf. Section III-B). Our method works for any dimension and for any dilation, but it would be most beneficial for multi-D case since this is where the construction gets more difficult. The method provides algorithms for constructing multi-D wavelet FBs from a single lowpass filter with minimal assumptions: positive accuracy and unimodularity of the polyphase representation (cf. Section III-C).

Our findings in this paper show that constructing multi-D wavelet FBs using the Quillen-Suslin Theorem, a well-known result in Algebraic Geometry, offers some noteworthy advantages over other more traditional approaches. We plan to explore the opportunities to study other challenges in multi-D wavelet FB construction using Algebraic Geometry techniques in our future researches.

Appendix

A. Proof of Corollary 1

We first recall that a filter f has accuracy number $k \in \mathbb{N}$ if and only if its Fourier transform $F(e^{i\omega})$ satisfies

$$F(e^{i(\omega+\gamma)}) = O(|\omega|^k), \quad (\text{near } \omega = 0),$$

for all $\gamma \in \Gamma^*\backslash\{0\}$, and it has flatness $k \in \mathbb{N}$ if and only if

$$\sqrt{q} - F(e^{i\omega}) = O(|\omega|^k), \quad (\text{near } \omega = 0).$$

From the proof of Theorem 1, we know that for any lowpass filters h, f, and g that satisfy the assumptions of Corollary 1, there exists a dual lowpass filter d of h whose polyphase representation satisfies

$$\mathtt{D}(z) = \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z)).$$

The z-transform of d is obtained via

$$\begin{aligned} D(z) =& v(z)^* \mathtt{D}(z^\Lambda) \\ =& v(z)^*\mathtt{G}(z^\Lambda) + v(z)^*\mathtt{F}(z^\Lambda)(1 - \mathtt{H}(z^\Lambda)\mathtt{G}(z^\Lambda)) \\ =& G(z) + F(z)(1 - \mathtt{H}(z^\Lambda)\mathtt{G}(z^\Lambda)) \\ =& G(z) + F(z)B(z^\Lambda) \end{aligned}$$

where $B(z) := 1 - \mathtt{H}(z)\mathtt{G}(z)$. Let $z = e^{i(\omega+\gamma)}$, then

$$D(e^{i(\omega+\gamma)}) = G(e^{i(\omega+\gamma)}) + F(e^{i(\omega+\gamma)})B((e^{i(\omega+\gamma)})^\Lambda).$$

Thus it suffices to show that

$$D(e^{i(\omega+\gamma)}) = O(|\omega|^{\min\{\alpha_g, \alpha_f+\beta_g, \alpha_f+\beta_h\}})$$

near $\omega = 0$, for all $\gamma \in \Gamma^*\backslash\{0\}$.

From the fact that $B((e^{i(\omega+\gamma)})^\Lambda) = B((e^{i\omega})^\Lambda)$ for all $\gamma \in \Gamma^*\backslash\{0\}$, and the simple observation (cf. Appendix C in [26])

$$B((e^{i\omega})^\Lambda) = 1 - \frac{1}{q}\sum_{\gamma\in\Gamma^*} H(e^{i(\omega+\gamma)})G(e^{i(\omega+\gamma)}),$$

we have $B((e^{i\omega})^\Lambda)$

$$\begin{aligned} =& 1 - \frac{1}{q}H(e^{i\omega})G(e^{i\omega}) + O(|\omega|^{\alpha_h+\alpha_g}) \\ =& 1 - \frac{1}{q}(\sqrt{q} + O(|\omega|^{\beta_h}))(\sqrt{q} + O(|\omega|^{\beta_g})) + O(|\omega|^{\alpha_h+\alpha_g}) \\ =& O(|\omega|^{\min\{\beta_h,\beta_g,\alpha_h+\alpha_g\}}), \quad (\text{near } \omega = 0). \end{aligned}$$

Therefore

$$\begin{aligned} D(e^{i(\omega+\gamma)}) =& G(e^{i(\omega+\gamma)}) + F(e^{i(\omega+\gamma)})B((e^{i(\omega+\gamma)})^\Lambda) \\ =& O(|\omega|^{\alpha_g}) + O(|\omega|^{\alpha_f})O(|\omega|^{\min\{\beta_h,\beta_g,\alpha_h+\alpha_g\}}) \\ =& O(|\omega|^{\min\{\alpha_g,\alpha_f+\beta_g,\alpha_f+\beta_h\}}) \end{aligned}$$

near $\omega = 0$, for all $\gamma \in \Gamma^*\backslash\{0\}$. ■

B. Proof of Corollary 2

In this proof, we use an iterative method to construct a dual lowpass filter d of h such that the accuracy number of d is at least α_h. For any lowpass filters h with positive accuracy α_h, if we let $g := h$ and f be a dual lowpass filter of h, then by Corollary 1 and its proof, we know that there exists a dual lowpass filter d of h whose polyphase representation is

$$\mathtt{D}(z) = \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z)) \tag{5}$$

and its accuracy number is at least $\min\{\alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\} = \min\{\alpha_h, \alpha_f + \beta_h\}$. If $\alpha_f + \beta_h < \alpha_h$, then we set $f := d$, and use this new f in (5) to construct a new d. This new d now has accuracy number at least $\min\{\alpha_h, \alpha_f + 2\beta_h\}$. Since $\beta_h \geq 1$, $\alpha_f + 2\beta_h$ is strictly larger that $\alpha_f + \beta_h$, and if $\alpha_f + 2\beta_h < \alpha_h$, we can iteratively update f to be the new d until $\alpha_f + (Ite)\beta_h \geq \alpha_h$, where Ite denotes the number of iterations. Thus we obtain a dual lowpass filter d whose accuracy number is at least α_h. ■

C. Proof of Corollary 3

Since $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, \cdots, 0]^T$, we have $\mathtt{H}(z)\mathtt{K}(z)^T = [1, 0, \cdots, 0]$. Therefore, $\mathtt{H}(z) = [1, 0, \cdots, 0][\mathtt{K}(z)^T]^{-1}$, i.e., the first row of $[\mathtt{K}(z)^T]^{-1}$ is $\mathtt{H}(z)$.

Let $\mathtt{T}(z)$ in the proof of Theorem 1 be $[\mathtt{K}(z)^T]^{-1}$. Then $\mathtt{H}^u(z) = \mathtt{H}(z)[\mathtt{T}(z)]^{-1} = [1, 0, \cdots, 0]$ and the first component of $\mathtt{F}^w(z) = \mathtt{T}(z)\mathtt{F}(z)$ is 1 since the first row of $[\mathtt{K}(z)^T]^{-1}$ is $\mathtt{H}(z)$ and f is dual to h.

Then, after some calculation, we see that the product of the first two matrices on the left-hand side of identity (3) in the proof of Theorem 1 becomes

$$\begin{bmatrix} & 0 & 0 & \cdots & 0 \\ & 0 & 1 & & \\ \mathtt{D}^w(z) & \vdots & & \ddots & \\ & 0 & & & 1 \end{bmatrix},$$

where $\mathtt{D}^w(z) := \mathtt{G}^w(z) + \mathtt{F}^w(z)(1 - \mathtt{H}^u(z)\mathtt{G}^w(z))$ and $c(z)$ in the reduction matrix $\mathtt{R}(z)$ is taken to be 1. The product of the last two matrices on the left-hand side of identity (3) becomes

$$\begin{bmatrix} & \mathtt{H}^u(z) & & \\ 1 - G^w_{\nu_0}(z) & 0 & \cdots & 0 \\ -G^w_{\nu_1}(z) - F^w_{\nu_1}(z)(1 - G^w_{\nu_0}(z)) & 1 & & \\ \vdots & & \ddots & \\ -G^w_{\nu_{q-1}}(z) - F^w_{\nu_{q-1}}(z)(1 - G^w_{\nu_0}(z)) & & & 1 \end{bmatrix}.$$

By deleting the second column of the first matrix and the second row of the second matrix in the above equation, we obtain a non-redundant FB. From Theorem 1, we know that this FB is a wavelet FB. The analysis lowpass filter is h and the synthesis lowpass filter d has polyphase representation $\mathtt{D}(z) = \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$. From Corollary 1, we know that the accuracy number of d is at least $\min\{\alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$. Therefore this wavelet FB has at least $\min\{\alpha_h, \alpha_g, \alpha_f + \beta_g, \alpha_f + \beta_h\}$ vanishing moments.

Let $k_1, \cdots, k_{q-1}$ be the synthesis highpass filters and $j_1, \cdots, j_{q-1}$ be the analysis highpass filters of the non-redundant wavelet FB that we just found. Let $e_0 := [1, 0, \cdots, 0]^T, e_1 = [0, 1, 0, \cdots, 0]^T, \cdots, e_{q-1} = [0, 0, \cdots, 0, 1]^T$ be the standard unit vectors in $\mathbb{R}^q$. Then from the synthesis side (the one derived from the first matrix of the above matrix identity) of the non-redundant wavelet FB, we see that the polyphase representation for the synthesis highpass filter k_i, for $i = 1, \cdots, q-1$, is

$$[\mathtt{T}(z)]^{-1} e_i = \mathtt{K}(z)^T e_i = (i+1)\text{th column of } \mathtt{K}(z)^T.$$

The polyphase representation for the analysis highpass filter j_i, for $i = 1, \cdots, q-1$, can be obtained from the analysis side (the one derived from the second matrix of the above matrix identity) of the non-redundant wavelet FB. They are

$$\begin{aligned}
&\Big[\Big(-G^w_{\nu_i}(z) - F^w_{\nu_i}(z)\big(1 - G^w_{\nu_0}(z)\big)\Big)e_0^T + e_i^T\Big]\mathtt{T}(z) \\
&= \Big[\Big(-e_i^T\mathtt{T}(z)\mathtt{G}(z) - e_i^T\mathtt{T}(z)\mathtt{F}(z)\big(1 - e_0^T\mathtt{T}(z)\mathtt{G}(z)\big)\Big)e_0^T \\
&\quad + e_i^T\Big]\mathtt{T}(z) \\
&= e_i^T\mathtt{T}(z)\Big[-\mathtt{G}(z)e_0^T\mathtt{T}(z) - \mathtt{F}(z)e_0^T\mathtt{T}(z) \\
&\quad + \mathtt{F}(z)e_0^T\mathtt{T}(z)\mathtt{G}(z)e_0^T\mathtt{T}(z) + \mathtt{I}_q\Big] \\
&= e_i^T\mathtt{T}(z)\Big[-\mathtt{G}(z)\mathtt{H}(z) - \mathtt{F}(z)\mathtt{H}(z) + \mathtt{F}(z)\mathtt{H}(z)\mathtt{G}(z)\mathtt{H}(z) + \mathtt{I}_q\Big] \\
&= e_i^T[\mathtt{K}(z)^T]^{-1}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)] \\
&= (i+1)\text{th row of } [\mathtt{K}(z)^T]^{-1}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)],
\end{aligned}$$

for $i = 1, \cdots, q-1$, and this concludes the proof. ■

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