The Design of Non-redundant Directional Wavelet Filter Bank Using 1-D Neville Filters

Youngmi Hur, Fang Zheng
Department of Applied Mathematics and Statistics
Johns Hopkins University, Baltimore, Maryland 21218
Email: hur@jhu.edu, fzheng2@jhu.edu

***Abstract*—In this paper, we develop a method to construct non-redundant directional wavelet filter banks. Our method uses a special class of filters called Neville filters and can construct non-redundant wavelet filter banks in any dimension for any dilation matrix. The resulting filter banks have directional analysis highpass filters, thus can be used in extracting directional contents in multi-D signals such as images. Furthermore, one can custom-design the directions of highpass filters in the filter banks.**

I. Introduction

In the last couple of decades, wavelets have been a popular and useful tool in many applications such as signal and image processing. One of important remaining challenges in wavelets is to construct multi-D directional wavelet systems or wavelet filter banks.

There has been a lot of attempts to develop such wavelet systems or their variants for 2-D or 3-D signals, such as curvelets, contourlets, shearlets, etc. Despite many benefits of these existing systems, most of them are redundant with possibly huge redundancy factors, and they do not have a trivial generalization to higher dimensions. Although a recent study by the authors provides the construction of non-redundant wavelet filter banks with directional highpass filters for any dimension [1], it only deals with the dyadic dilation matrices. Other approaches based on anisotropic wavelet bases have also been proposed (see, for example, [2], [3], [4] and the references therein). However, these wavelets are designed in continuous domain and implementing them in discrete setting is not trivial.

In this paper, we develop a new method to construct non-redundant wavelet filter banks that can capture the directional information in multi-D signals. Our method is a general designing recipe in the sense that it can work in any dimension for any dilation matrix. In the design, one can even specify the number of directions and which directions to consider.

II. Preliminaries

In this section, we review some basic concepts and notations about wavelet filter bank construction. In particular, we review the concept of Neville filters and how to use Neville filters to build multi-D wavelet filter banks.

This work was supported in part by the National Science Foundation under Grant DMS-1115870.

A. Notation

In this paper, we use boldface to indicate vectors and matrices. A filter f is a a linear time-invariant operator characterized by its impulse response $\{f(\mathbf{k}) \in \mathbb{R} | \mathbf{k} \in \mathbb{Z}^d\}$. The z-transform of a filter is a Laurent polynomial

$$F(\mathbf{z}) = \sum_{\mathbf{k}} f(\mathbf{k})\mathbf{z}^{-\mathbf{k}}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and $\mathbf{z}^{\mathbf{k}} := \prod_{i=1}^d z_i^{k_i}$. In this paper, we refer to both the z-transform $F(\mathbf{z})$ and the impulse response $f(\mathbf{k})$ as the filter, and sometimes we omit $\mathbf{z}$ and $\mathbf{k}$ in the parentheses for convenience. Define the *adjoint* of a filter as $[F(\mathbf{z})]^* := F(1/\mathbf{z})$. Throughout this paper, we assume all filters have finite impulse response.

A dilation matrix $\mathbf{D}$ is a $d \times d$ integer matrix with $|\det \mathbf{D}| := m > 1$. Given a dilation matrix $\mathbf{D}$, the set $\mathbb{Z}^d$ of integer grids can be split into m disjoint subsets

$$\mathbb{Z}^d = \bigcup_{i=0}^{m-1} (\mathbf{D}\mathbb{Z}^d + \mathbf{t}_i), \quad \mathbf{t}_i \in \mathbb{Z}^d$$

where $\mathbf{t}_0 = \mathbf{0}$. We call $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{m-1}\}$ as a set of *(nonzero) distinct coset representatives* of the dilation matrix $\mathbf{D}$.

A filter bank (FB) consisting of an analysis bank and a synthesis bank is a set of filters. For a given dilation matrix $\mathbf{D}$, a filter in the analysis bank $\{A_i, i = 0, \ldots, l-1\}$ and a filter in the synthesis bank $\{S_i, i = 0, \ldots, l-1\}$ can be written as the sum of m *polyphase components*

$$A_i(\mathbf{z}) = \sum_{j=0}^{m-1} \mathbf{z}^{\mathbf{t}_j} A_{i,j}(\mathbf{z}^{\mathbf{D}}), \quad a_{i,j}(\mathbf{k}) := a_i(\mathbf{D}\mathbf{k} - \mathbf{t}_j) \quad (1)$$

$$S_i(\mathbf{z}) = \sum_{j=0}^{m-1} \mathbf{z}^{-\mathbf{t}_j} S_{i,j}(\mathbf{z}^{\mathbf{D}}), \quad s_{i,j}(\mathbf{k}) := s_i(\mathbf{D}\mathbf{k} + \mathbf{t}_j) \quad (2)$$

where $\mathbf{z}^{\mathbf{D}} := (\mathbf{z}^{D_1}, \mathbf{z}^{D_2}, \ldots, \mathbf{z}^{D_d})$, D_i is the ith column vector of $\mathbf{D}$. Then the pair of matrices

$$\begin{aligned} \mathbf{A}(\mathbf{z}) &:= [A_{i,j}(\mathbf{z})]_{i=0,\ldots,l-1; j=0,\ldots,m-1} \\ \mathbf{S}(\mathbf{z}) &:= [S_{j,i}(\mathbf{z})]_{j=0,\ldots,m-1; i=0,\ldots,l-1} \end{aligned}$$

is called the *polyphase matrix representation* [5] of the FB.

A FB satisfies the *perfect reconstruction* condition if the polyphase matrices satisfy $\mathbf{S}(\mathbf{z})\mathbf{A}(\mathbf{z}) = \mathbf{I}_m$, which can happen only when $l \geq m$. A FB is called *non-redundant* if $l = m$.

In this paper, we are only interested in non-redundant FBs satisfying the perfect reconstruction condition, and we assume there are exactly one lowpass filter A_0 in the analysis bank and one lowpass filter S_0 in the synthesis bank. The rest, $A_1, \ldots, A_{m-1}, S_1, \ldots, S_{m-1}$, are all highpass filters.

We use Π_N to denote the set of all polynomials of total degree less than N. We say a FB has $N \in \mathbb{N}$ *vanishing moments* [6] if, for any highpass filter f in the FB, $(f *' \pi)(\mathbb{Z}^d) = 0, \forall \pi \in \Pi_N$, or equivalently,

$$\textstyle\sum_{\mathbf{k}} f(-\mathbf{k})\mathbf{k}^{\mathbf{n}} = 0,\ \forall \mathbf{n} \in \mathbb{N}_0^d,\ |\mathbf{n}| < N$$

where $\mathbf{n} := (n_1, n_2, \ldots, n_d)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $|\mathbf{n}| := n_1 + n_2 + \ldots + n_d$. Here we used $(f *' \pi)(\cdot) := \sum_{\mathbf{k} \in \mathbb{Z}^d} f(\mathbf{k})\pi(\cdot - \mathbf{k})$.

B. Neville Filters and Their Use in Wavelet FB Construction

In [7], Kovačević and Sweldens introduce a class of filters called *Neville filters* (Definition 1) and their characterization (Result 1). When applied to a sampled polynomial, they result in the same polynomial but shifted by a shift parameter $\boldsymbol{\tau} \in \mathbb{R}^d$.

Definition 1. A filter f is a Neville filter of order N with shift $\boldsymbol{\tau}$ if $(f *' \pi)(\mathbb{Z}^d) = \pi(\mathbb{Z}^d + \boldsymbol{\tau})$, for any $\pi \in \Pi_N$. ∎

Result 1 (Proposition 4 in [7]). A filter f is a Neville filter of order N with shift $\boldsymbol{\tau}$ if and only if f satisfies

$$\textstyle\sum_{\mathbf{k}} f(-\mathbf{k})\mathbf{k}^{\mathbf{n}} = \boldsymbol{\tau}^{\mathbf{n}},\ \forall \mathbf{n} \in \mathbb{N}_0^d,\ |\mathbf{n}| < N. \tag{3}$$

∎

In 1-D case, the construction of Neville filters of order N is straightforward. Once we fix the positions of N filter taps, we obtain a linear system with an $N \times N$ coefficient matrix from (3). Since the coefficient matrix in this case is a Vandermonde matrix, it is always solvable. In multi-D case, the solvability of the linear system not only depends on the number of filter taps but also on the geometric shape of the filter. Hence it is more challenging to construct a multi-D Neville filter with a prescribed order and shift. An approach based on an algorithm in [8] to solve this problem is proposed in [7], but it is highly non-trivial to control the shape of the filters using that approach.

Using the property of Neville filters, Kovačević and Sweldens propose a method for constructing wavelet FBs based on lifting scheme [9]. They use two lifting steps: predict (cf. R_i) and update (cf. U_i), as shown in (4) and (5) to build the wavelet FB with desirable vanishing moments:

$$\mathbf{A} = \begin{bmatrix} 1 & U_1 & \cdots & U_{m-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -R_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R_{m-1} & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \sum_{i=1}^{m-1} U_i R_i & U_1 & \cdots & U_{m-1} \\ -R_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R_{m-1} & 0 & \cdots & 1 \end{bmatrix} \tag{4}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ R_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{m-1} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & -U_1 & \cdots & -U_{m-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -U_1 & \cdots & -U_{m-1} \\ R_1 & 1 - R_1 U_1 & \cdots & -R_1 U_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m-1} & -R_{m-1} U_1 & \cdots & 1 - R_{m-1} U_{m-1} \end{bmatrix}, \tag{5}$$

where R_i are called predict filters, U_i are called update filters, and $m = |\det \mathbf{D}|$. More precisely, the following is a variant of the result they prove in [7], written in terms of our terminology.

Result 2. Let $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{m-1}\}$ be a set of distinct coset representatives of the $d \times d$ dilation matrix $\mathbf{D}$. For $i = 1, \cdots, m-1$, let R_i be a d-D Neville filter of order N with shift $\boldsymbol{\tau}_i = \mathbf{D}^{-1}\mathbf{t}_i$, and U_i be the filter obtained by multiplying $1/m$ to the adjoint of a d-D Neville filter of order N with shifts $\boldsymbol{\tau}_i$. Then the analysis polyphase matrix constructed as (4) and the synthesis polyphase matrix constructed as (5) form a wavelet FB with N vanishing moments. ∎

This construction works for any dilation matrix $\mathbf{D}$ in any dimension. It uses d-D Neville filters with prescribed orders and shifts to construct d-D wavelet FBs.

III. Directional wavelet FB design using 1-D Neville filters

In this section, we introduce a method to design directional wavelet FBs using 1-D Neville filters and the lifting based wavelet construction method reviewed in Section II-B. Let us first define an operator that maps 1-D filters to d-D filters.

Definition 2. Define the operator that maps a 1-D filter F to a d-D filter $\mathcal{M}_{\mathbf{t}}(F)$ along direction $\mathbf{t} \in \mathbb{Z}^d$ as

$$\mathcal{M}_{\mathbf{t}}(F)(\mathbf{z}) := F(\mathbf{z}^{\mathbf{t}}).$$

∎

The following simple lemma, which says that the operator $\mathcal{M}_{\mathbf{t}}$ preserves the order of Neville filters is a key ingredient of our directional wavelet FB construction.

Lemma 1. *If F is a 1-D Neville filter of order N with shift $\tau \in \mathbb{R}$, then the d-D filter $\mathcal{M}_{\mathbf{t}}(F)$ is a Neville filter of order N with shift $\tau\mathbf{t}, \mathbf{t} \in \mathbb{Z}^d$.*

Proof: Let $G := \mathcal{M}_{\mathbf{t}}(F)$, and let g be the impulse response of G. Then, we have

$$g(\mathbf{k}) = \begin{cases} f(k), & \text{if } \mathbf{k} = k\mathbf{t} \text{ for some } k \in \mathbb{Z}\ , \\ 0, & \text{for all other } \mathbf{k} \in \mathbb{Z}^d. \end{cases}$$

where f is the impulse response of F. Therefore

$$\begin{aligned} \textstyle\sum_{\mathbf{k}} g(-\mathbf{k})\mathbf{k}^{\mathbf{n}} &= \textstyle\sum_k f(-k)(k\mathbf{t})^{\mathbf{n}} = \sum_k f(-k)k^{|\mathbf{n}|}\mathbf{t}^{\mathbf{n}} \\ &= \tau^{|\mathbf{n}|}\mathbf{t}^{\mathbf{n}} = (\tau\mathbf{t})^{\mathbf{n}}, \end{aligned}$$

for any $\mathbf{n} \in \mathbb{N}_0^d$, $|\mathbf{n}| < N$, where the second last equation holds because F is a 1-D Neville filter of order N with shift τ. Thus G is a d-D Neville filter of order N with shift $\tau\mathbf{t}$. ∎

Example 1: Mapping 1-D Neville Filter to 2-D. $F(z) = 1/3z + 2/3$ is a 1-D Neville filter of order 2 with shift $\tau = 1/3$. Then mapping it to 2-D along direction $\mathbf{t} = (1, 1)$ results in $\mathcal{M}_{\mathbf{t}}(F)(\mathbf{z}) = 1/3z_1z_2 + 2/3$. It can be easily checked that $\mathcal{M}_{\mathbf{t}}(F)$ is a Neville filter of order 2 with shift $\tau\mathbf{t} = (1/3, 1/3)$. Figure 1 shows the impulse response of F and $\mathcal{M}_{\mathbf{t}}(F)$. ∎

From Example 1, we see that the multi-D Neville filter constructed by the operator $\mathcal{M}_{\mathbf{t}}$ is directional along direction $\mathbf{t}$. We now discuss how to use these directional multi-D Neville filters to construct directional wavelet FB.

$$\begin{array}{cc} \frac{1}{3} & \underline{\frac{2}{3}} \end{array} \longrightarrow \begin{array}{cc} 0 & \frac{2}{3} \\ \underline{\frac{1}{3}} & 0 \end{array}$$

Fig. 1. Mapping 1-D Neville filter to 2-D. The impulse response of F and $\mathcal{M}_{\mathbf{t}}(F)$ in Example 1. Underlined position is the origin.

Let us first look at a simple case when the dilation matrix $\mathbf{D} = c\mathbf{I}_d$ where $c \in \mathbb{Z},\ c > 1$ and $\mathbf{I}_d$ is the identity matrix. In this case, $\mathbf{D}^{-1} = (1/c)\mathbf{I}_d$. The multi-D Neville filters used to construct predict and update filters in Result 2 need to have shift parameters $\boldsymbol{\tau}_i = \mathbf{D}^{-1}\mathbf{t}_i = (1/c)\mathbf{t}_i$. Therefore, it is possible to construct all these multi-D Neville filters by mapping a single 1-D Neville filter with shift $\tau = 1/c$ but with different directions $\mathbf{t}_i$. In this way, we can avoid constructing multi-D Neville filters directly, which is often difficult to do. Moreover, it can be shown that the highpass filters built on these multi-D Neville filters are also directional.

To generalize this idea to a general dilation matrix $\mathbf{D}$, let us consider the shift parameters $\boldsymbol{\tau}_i = \mathbf{D}^{-1}\mathbf{t}_i$ again. In this case, if we factor out $\tau = 1/m$ as the shift parameter for 1-D Neville filters, then $\boldsymbol{\tau}_i = \tau\tilde{\mathbf{t}}_i$, where $\tilde{\mathbf{t}}_i = m\mathbf{D}^{-1}\mathbf{t}_i \in \mathbb{Z}^d$, hence we can map a single 1-D Neville filter with shift $\tau = 1/m$ along different directions $\tilde{\mathbf{t}}_i$. For example, for dilation matrix

$$\mathbf{D} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \tag{6}$$

a set of distinct coset representatives of $\mathbf{D}$ are $\mathbf{t}_1 = (0,1), \mathbf{t}_2 = (1,1), \mathbf{t}_3 = (0,2), \mathbf{t}_4 = (1,2)$. The shift parameters of Neville filters needed to construct wavelet FB are $\boldsymbol{\tau}_1 = (1/5, 2/5), \boldsymbol{\tau}_2 = (3/5, 1/5), \boldsymbol{\tau}_3 = (2/5, 4/5), \boldsymbol{\tau}_4 = (4/5, 3/5)$. Therefore, we can construct all these multi-D Neville filters by mapping one 1-D Neville filter with shift $1/5$ along directions $\tilde{\mathbf{t}}_1 = (1,2), \tilde{\mathbf{t}}_2 = (3,1), \tilde{\mathbf{t}}_3 = (2,4), \tilde{\mathbf{t}}_4 = (4,3)$.

In fact, we can factor out any $\tau = 1/s$, where $s \in \mathbb{Z}$, as the shift parameter for 1-D Neville filters, as long as $\boldsymbol{\tau}_i = \tau\tilde{\mathbf{t}}_i$ and $\tilde{\mathbf{t}}_i = s\mathbf{D}^{-1}\mathbf{t}_i \in \mathbb{Z}^d$. In the simple case when $\mathbf{D} = c\mathbf{I}_d$, $s := c$ can be chosen, while in other cases such as (6), $s := m$ can be chosen. Therefore, we have the following theorem. For a general d-D dilation matrix $\mathbf{D}$ with $|\det\mathbf{D}| = m$, we can construct a directional wavelet FB with analysis highpass filters presenting at most $m-1$ different directions as follows.

Theorem 1. *Let $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{m-1}\}$ be a set of distinct coset representatives of $\mathbf{D}$. Let s be an integer such that $s\mathbf{D}^{-1}\mathbf{t}_i \in \mathbb{Z}^d$. For $i = 1, \cdots, m-1$, let P_i and Q_i be the 1-D Neville filters of order N with shift $1/s$. Set $\tilde{\mathbf{t}}_i = s\mathbf{D}^{-1}\mathbf{t}_i$. Let d-D filter $R_i := \mathcal{M}_{\tilde{\mathbf{t}}_i}(P_i)$ and $U_i := (1/m)[\mathcal{M}_{\tilde{\mathbf{t}}_i}(Q_i)]^*$. Then the analysis polyphase matrix given by (4) and the synthesis polyphase matrix given by (5) form a directional FB with N vanishing moments and the analysis highpass filters are placed along directions $\mathbf{t}_i$.* ∎

Proof: Since P_i (resp. Q_i) is a 1-D Neville filter of order N with shift $1/s$, by Lemma 1, $R_i = \mathcal{M}_{\tilde{\mathbf{t}}_i}(P_i)$ (resp. $\mathcal{M}_{\tilde{\mathbf{t}}_i}(Q_i)$) is a d-D Neville filter of order N with shift $(1/s)\tilde{\mathbf{t}}_i = (1/s)s\mathbf{D}^{-1}\mathbf{t}_i = \mathbf{D}^{-1}\mathbf{t}_i$. Thus $U_i = (1/m)[\mathcal{M}_{\tilde{\mathbf{t}}_i}(Q_i)]^*$ is $1/m$ times the adjoint of Neville filter of order N with shift $\mathbf{D}^{-1}\mathbf{t}_i$. By Result 2, we see that (4) and (5) form a wavelet FB with N vanishing moments.

To prove the directionality of analysis highpass filters, consider the ith analysis highpass filter denoted by A_i. Since

$$R_i(\mathbf{z}) = \mathcal{M}_{\tilde{\mathbf{t}}_i}(P_i)(\mathbf{z}) = P(\mathbf{z}^{\tilde{\mathbf{t}}_i}) = P(\mathbf{z}^{s\mathbf{D}^{-1}\mathbf{t}_i}),$$

from (1) and (4), we see that $A_i(\mathbf{z})$ is equal to

$$-R_i(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{\mathbf{t}_i} = -P_i(\mathbf{z}^{\mathbf{D}s\mathbf{D}^{-1}\mathbf{t}_i}) + \mathbf{z}^{\mathbf{t}_i} = -P_i(\mathbf{z}^{s\mathbf{t}_i}) + \mathbf{z}^{\mathbf{t}_i}.$$

If we replace $\mathbf{z}^{\mathbf{t}_i}$ with z in the last equation on the right hand side, we get a 1-D filter $-P_i(z^s) + z$. Thus A_i can be understood as the result of taking the 1-D filter $-P_i(z^s) + z$ and placing it in d-D space along direction $\mathbf{t}_i$. ∎

Remark 1. In Theorem 1, a single 1-D Neville filter of order N and shift $1/m$ can be used for all of P_i and Q_i, or different 1-D Neville filters can be used. In fact P_i and Q_i can have different orders if we invoke more generalized version of Result 2 from [7]. In this case, if P_i's order is $\tilde{N}_i$ and Q_i's order is N_i, then the vanishing moments of the FB is given as $\min\{\tilde{N}_1, \ldots, \tilde{N}_{m-1}, N_1, \ldots, N_{m-1}\}$.

Remark 2. The analysis highpass filters A_i of the FB in Theorem 1 are placed along directions $\mathbf{t}_i \in \mathbb{Z}^d, i = 1, \ldots, m-1$ (not $\tilde{\mathbf{t}}_i = m\mathbf{D}^{-1}\mathbf{t}_i$). Therefore, by carefully choosing the distinct coset representatives of $\mathbf{D}$, one can custom-design the directions of the filters (cf. Example 2). There are *at most* $m-1$ different directions that can be presented by the analysis highpass filters.

In the next example, we illustrate how to use Theorem 1 to construct directional wavelet FB.

Example 2: 2-D Directional Wavelet FB with 2 Vanishing Moments. For dilation matrix $\mathbf{D} = 3\mathbf{I}_2$, since $|\det\mathbf{D}| = 9$, there are $9 - 1 = 8$ distinct coset representatives $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_8\}$ that we can choose. We know that the directions of coset representatives are exactly the directions of resulting analysis highpass filters. Here we want to choose directions that divide the 2-D plane as equally as possible. Thus we choose $\mathbf{t}_1 = (1,0), \mathbf{t}_2 = (-1,0), \mathbf{t}_3 = (0,1), \mathbf{t}_4 = (0,-1), \mathbf{t}_5 = (2,1), \mathbf{t}_6 = (1,2), \mathbf{t}_7 = (-2,1), \mathbf{t}_8 = (-1,2)$. Then the resulting analysis highpass filters will present 6 different directions in the 2-D plane: approximately, $0°$ ($\mathbf{t}_1$, $\mathbf{t}_2$), $30°$ ($\mathbf{t}_5$), $60°$ ($\mathbf{t}_6$), $90°$ ($\mathbf{t}_3$, $\mathbf{t}_4$), $120°$ ($\mathbf{t}_8$) and $150°$ ($\mathbf{t}_7$) from the positive x-axis.

Next we pick a single 1-D Neville filter of order 2 with shift $1/3$ for all P_i and Q_i: $P_i(z) = Q_i(z) = 1/3z + 2/3$, for $i = 1, \ldots, 8$. Theorem 1 says that if we choose, for each i,

$$\begin{aligned} R_i(\mathbf{z}) &= P_i(\mathbf{z}^{\mathbf{t}_i}) = 1/3\mathbf{z}^{\mathbf{t}_i} + 2/3 \\ U_i(\mathbf{z}) &= (1/m)[Q_i(\mathbf{z}^{\mathbf{t}_i})]^* = (1/9)(1/3\mathbf{z}^{-\mathbf{t}_i} + 2/3) \end{aligned}$$

then we get the wavelet FB with 2 vanishing moments, whose polyphase matrices are $\mathbf{A}$ and $\mathbf{S}$ in (4) and (5). Using formula (1) and (2), we can read off the corresponding filters. For example, the resulting synthesis lowpass filter S_0 is

$$S_0(\mathbf{z}) = 1 + \textstyle\sum_{i=1}^{8} \mathbf{z}^{-\mathbf{t}_i} R_i(\mathbf{z}^{\mathbf{D}})$$

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0\\
0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0\\
0 & 0 & \frac{1}{3} & \frac{2}{3} & \underline{1} & \frac{2}{3} & \frac{1}{3} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0\\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0
\end{array}$$

(a) Synthesis lowpass filter S_0

$$\begin{array}{cccc} -\frac{1}{3} & 0 & 1 & \underline{-\frac{2}{3}} \end{array}$$

(b) $A_1 : \mathbf{t}_1 = (1, 0)$

$$\begin{array}{cccc} \underline{-\frac{2}{3}} & 1 & 0 & -\frac{1}{3} \end{array}$$

(c) $A_2 : \mathbf{t}_2 = (-1, 0)$

$$\begin{array}{c} \underline{-\frac{2}{3}} \\ 1 \\ 0 \\ -\frac{1}{3} \end{array}$$

(d) $A_3 : \mathbf{t}_3 = (0, 1)$

$$\begin{array}{c} -\frac{1}{3} \\ 0 \\ 1 \\ \underline{-\frac{2}{3}} \end{array}$$

(e) $A_4 : \mathbf{t}_4 = (0, -1)$

$$\begin{array}{ccccccc}
 & & & & 0 & & \underline{-\frac{2}{3}}\\
0 & 0 & 0 & 0 & 1 & 0 & 0\\
 & & & & 0 & & \\
-\frac{1}{3} & & & & 0 & &
\end{array}$$

(f) $A_5 : \mathbf{t}_5 = (2, 1)$

$$\begin{array}{cccc}
 & & 0 & \underline{-\frac{2}{3}}\\
 & & 0 & \\
0 & 0 & 1 & 0\\
 & & 0 & \\
 & & 0 & \\
 & & 0 & \\
-\frac{1}{3} & & 0 &
\end{array}$$

(g) $A_6 : \mathbf{t}_6 = (1, 2)$

$$\begin{array}{ccccccc}
\underline{-\frac{2}{3}} & & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
 & & 0 & & & & \\
 & & 0 & & & & -\frac{1}{3}
\end{array}$$

(h) $A_7 : \mathbf{t}_7 = (-2, 1)$

$$\begin{array}{cccc}
\underline{-\frac{2}{3}} & 0 & & \\
 & 0 & & \\
0 & 1 & 0 & 0\\
 & 0 & & \\
 & 0 & & \\
 & 0 & & \\
 & 0 & & -\frac{1}{3}
\end{array}$$

(i) $A_8 : \mathbf{t}_8 = (-1, 2)$

Fig. 2. 2-D directional wavelet FB with 2 vanishing moments in Example 2: (a) synthesis lowpass filter, (b)-(i) directional analysis highpass filters with each direction along the coset representatives: $\mathbf{t}_i$, $i = 1, \ldots, 8$.

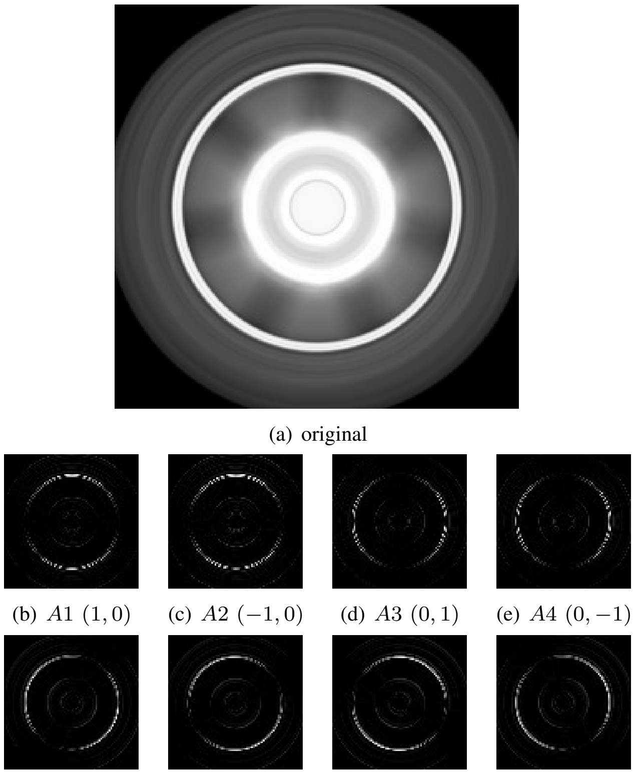
and the resulting analysis highpass filter associated with coset representative $\mathbf{t}_5 = (2, 1)$ is

$$A_5(\mathbf{z}) = -R_5(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{\mathbf{t}_5} = -(1/3z_1^6 z_2^3 + 2/3) + z_1^2 z_2.$$

Figure 2 shows the synthesis lowpass filter S_0 and the analysis highpass filters A_i, $i = 1, \ldots, 8$. ■

IV. Experimental Result

We did an experiment using the 2-D directional wavelet FB constructed in Example 2. For an original image "circle" (Figure 3(a)), we did a 1-level-down decomposition using the analysis highpass filters obtained in Example 2 (as shown in Figure 2(b)-(i)). The images after passing through each highpass filter (wavelet coefficients) are shown in Figure 3(b)-(i). The result shows that different directional components of the circle are captured by different directional highpass filters. A highpass filter with direction $\mathbf{t}$ can mainly capture the directional content that is orthogonal to the direction $\mathbf{t}$.



(a) original

(b) $A1$ $(1, 0)$ (c) $A2$ $(-1, 0)$ (d) $A3$ $(0, 1)$ (e) $A4$ $(0, -1)$

(f) $A5$ $(2, 1)$ (g) $A6$ $(1, 2)$ (h) $A7$ $(-2, 1)$ (i) $A8$ $(-1, 2)$

Fig. 3. (a) The original image "circle", (b)-(i) the images after passing highpass filters $A1, \ldots, A8$.

V. Conclusion

In this paper, we developed a method to use 1-D Neville filters to build multi-D directional wavelet FBs. The resulting FB is a non-redundant FB which can capture the directional information in multi-D signals.

References

[1] Y. Hur and F. Zheng, "Coset Sum: an alternative to the tensor product in wavelet construction," *IEEE Trans. Inform. Theory*, 2013, to be published.

[2] R. A. DeVore, S. V. Konyagin, and V. N. Temlyakov, "Hyperbolic wavelet approximation," *Constructive Approximation*, vol. 14, no. 1, pp. 1–26, 1998.

[3] J.-P. Antoine, P. Vandergheynst, and R. Murenzi, "Two-dimensional directional wavelets in image processing," *International journal of imaging systems and technology*, vol. 7, no. 3, pp. 152–165, 1996.

[4] H. Triebel, "Wavelet bases in anisotropic function spaces," in *Proc. Function Spaces, Differential Operators and Nonlinear Analysis (FSDONA2004)*, Milovy, Czech Republic, 2005, pp. 370–387.

[5] G. Strang and T. Nguyen, *Wavelets and Filter Banks.* Wellesley: Wellesley-Cambridge Press, 1997.

[6] D.-R. Chen, B. Han, and S. D. Riemenschneider, "Construction of multivariate biorthogonal wavelets with arbitrary vanishing moments," *Adv. Comput. Math.*, vol. 13, no. 2, pp. 131–165, 2000.

[7] J. Kovačević and W. Sweldens, "Wavelet families of increasing order in arbitrary dimensions," *Image Processing, IEEE Transactions on*, vol. 9, no. 3, pp. 480–496, 2000.

[8] C. de Boor and A. Ron, "On multivariate polynomial interpolation," *Constructive Approximation*, vol. 6, no. 3, pp. 287–302, 1990.

[9] W. Sweldens, "The lifting scheme: A custom-design construction of biorthogonal wavelets," *Appl. Comput. Harmon. Anal.*, vol. 3(2), pp. 186–200, 1996.