New Constructions of Nonseparable Tight Wavelet Frames

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Abstract

We present two methods for constructing new nonseparable multidimensional tight wavelet frames by combining the ideas of sum of squares representations of nonnegative trigonometric polynomials with the coset sum method of generating nonseparable multidimensional lowpass filters from univariate lowpass filters. In effect, these methods allow one to select a univariate lowpass filter and generate nonseparable multidimensional tight wavelet frames from it in any dimension $n \geq 2$, under certain conditions on the input filter which are given explicitly. We construct sum of hermitian squares representations for a particular class of trigonometric polynomials f in several variables, each related to a coset sum generated lowpass mask τ in that nonnegativity of f implies the sub-QMF condition for τ, in two ways: for interpolatory inputs to the coset sum method satisfying the univariate sub-QMF condition, we find this representation using the Fejér-Riesz Lemma; and in the general case, by writing $f = x^* P x$, where x is a vector of complex exponential functions, and P is a constant positive semidefinite matrix that is constructed to reduce the number of generators in this representation. The generators of this representation of f may then be used to generate the filters in a tight wavelet frame with lowpass mask τ. Several examples of these representations and the corresponding frames are given throughout.

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1. Introduction

The construction of wavelet systems with various special properties has been an area of active research for the past thirty years, and in recent years, has increasingly focused on multidimensional wavelet systems. Besides the standard tensor product method of creating a multidimensional wavelet system from a univariate system, it is much more difficult to create wavelet systems in multiple dimensions; however, it is well-known that there are desirable properties for wavelet systems in multiple dimensions which are not possible to enforce with the tensor product construction, and tensor product generated wavelets have been shown to be sub-optimal in certain applications (e.g., [8]).

Standard wavelet constructions are usually for *orthonormal* wavelet systems in $L_2(\mathbb{R}^n)$, in which the wavelet system forms an orthonormal basis for $L_2(\mathbb{R}^n)$, and analyzing and synthesizing a given function may be done using the same system. Common ways of increasing the flexibility in wavelet construction are to relax one or the other of these constraints: *biorthogonal* wavelet systems use different systems for the analysis and synthesis, but each system is still a basis for $L_2(\mathbb{R}^n)$, while wavelet *frames* are redundant in that they are not necessarily linearly independent sets in $L_2(\mathbb{R}^n)$. However, in the latter case, the analysis and synthesis may still be done by the same system if the wavelet frame is a *tight frame*. In addition to the added flexibility of frames, their redundancy makes them preferred in certain applications (see [11, 26] and references within). For additional information about frames of wavelets, see Section 2.2, which introduces some basic information, or for a more in-depth treatment, [6, 11].

An alternative method to the tensor product, which allows one to construct a nonseparable (i.e., non-tensor-product-based) multidimensional biorthogonal wavelet system, using a univariate biorthogonal wavelet system as input, is the coset sum [21]. In particular, the coset sum allows one to construct a nonseparable lowpass filter from a univariate lowpass filter[2], so in the present work, rather than constructing nonseparable biorthogonal systems, we construct nonseparable tight wavelet frames using lowpass filters generated from the coset sum method, combined with the ideas of sum of hermitian squares representations of nonnegative trigonometric polynomials. The connection between sum of hermitian squares (sos) representations and tight wavelet frames is given in [4] and [22], a review of which is found in Section 2.3; but the underlying idea for these methods is, for a lowpass mask τ, to represent the trigonometric polynomial $f(\tau;\omega) = 1 - 2^{-n} \sum_{\gamma\in\{0,\pi\}^n} |\tau(\omega/2 + \gamma)|^2$ for $\omega \in [-\pi, \pi]^n$, as a sum of hermitian squares. This representation of $f(\tau;\omega) = \sum_{1\leq j\leq M} |g_j(\omega)|^2$, $\omega \in [-\pi, \pi]^n$, for some trigonometric polynomials g_j, if it exists, is then used to find the frame generators with the lowpass mask τ.

In Section 2, we cover some preliminaries necessary for understanding our

[2] The coset sum method is not unique in this regard, since in [17], univariate B-spline filters are used to construct nonseparable multidimensional lowpass filters for general dilation matrices (though this is not the main contribution of that work).

methods including background about filters and masks in the context of wavelets, wavelet frames and the unitary extension principle, and sos representations and their connection to the construction of tight wavelet frames. In Section 3, we recall the coset sum method for constructing n-dimensional nonseparable lowpass masks from univariate masks, and introduce a few methods for constructing sos representations for $f(\tau;\omega)$ arising from coset sum generated lowpass masks τ, which in turn provide construction methods for tight wavelet frames with the lowpass mask τ. In Section 4, we collect several examples of our construction methods, and concluding remarks are given in Section 5.

2. Preliminaries

2.1. Filters and Masks in Wavelets

We say that τ, a trigonometric polynomial, is a *mask* associated with the *filter* $h : \mathbb{Z}^n \to \mathbb{R}$ (or $\mathbb{C}$, though we will assume that filters are real-valued in this work), which is only nonzero at finitely many points, if it is the Fourier transform of h; i.e., $\tau(\omega) = 2^{-n/2} \sum_{k\in\mathbb{Z}^n} h(k)e^{-ik\cdot\omega}$, with $\omega \in [-\pi, \pi]^n =: \mathbb{T}^n$, where n is the spatial dimension. We will reserve R and H as the symbols for mask and filter, respectively, in the case that $n = 1$, to differentiate between the input and output masks and filters for the coset sum operator below (see Section 3.1). When a statement is made for any dimension $n \geq 1$ (and especially if it is made for $n \geq 2$), we will use τ and h.

We say that a filter $h : \mathbb{Z}^n \to \mathbb{R}$ is *lowpass*, or *refinement*, if $\sum_{k\in\mathbb{Z}^n} h(k) = 2^n$, and if this sum is instead equal to 0, we say that the filter is *highpass*, or *wavelet*.

An important quantity associated with a lowpass mask is its *accuracy number*. In one dimension, this is defined as the order of the root that R has at π, i.e., if $R(\pi) = R'(\pi) = \cdots = R^{(m-1)}(\pi) = 0$, and $R^{(m)}(\pi) \neq 0$, we say that R has accuracy number m (and if R does not have a root at π, then it has accuracy number 0). In higher dimensions, the accuracy number is defined as the minimum order of the roots[3] that τ has at the points $\{0, \pi\}^n \setminus \{0\}$. The order of the root that a highpass mask has at 0 is called its number of *vanishing moments*.

Another important property of a lowpass mask is the *interpolatory property*. For $n \geq 1$, if a lowpass mask τ satisfies:

$$\sum_{\gamma\in\{0,\pi\}^n} \tau(\omega + \gamma) = 2^{n/2}, \quad \forall\omega \in \mathbb{T}^n$$

we say that τ has the interpolatory property, or is *interpolatory*. A wealth of additional information about wavelet transforms with interpolatory masks may be found in [12].

[3] For a particular $\gamma \in \{0, \pi\}^n \setminus \{0\}$, the order of the root that τ has at γ is the least $|\lambda| = \sum \lambda_i$ such that $(\partial_{\omega_1}^{\lambda_1} \cdots \partial_{\omega_n}^{\lambda_n} \tau)(\gamma) \neq 0$, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_i \geq 0$ for each $1 \leq i \leq n$.

For convenience, we will also sometimes say that a filter has a certain accuracy number or some other property of masks, and this should be understood to mean that the mask associated with this filter has the specified accuracy number or property. Similarly, if we say that a mask has a property usually associated with filters, this should be interpreted to mean that the filter associated with this mask has the stated property.

2.2. Wavelet Frames and the Unitary Extension Principle

For $r \geq 1$, we define the *multiresolution analysis-based (MRA-based) wavelet system with generators* $\psi^{(i)} \in L_2(\mathbb{R}^n)$, $1 \leq i \leq r$, *and refinable function* $\phi \in L_2(\mathbb{R}^n)$ as

$$\Lambda := \Lambda(\psi^{(1)}, \ldots, \psi^{(r)}) := \{\psi^{(i)}_{l,k} : 1 \leq i \leq r; l \in \mathbb{Z}, k \in \mathbb{Z}^n\}. \tag{1}$$

Here, the generators $\psi^{(i)}$ are called *mother wavelets* and defined via $\widehat{\psi^{(i)}}(\omega) = 2^{-n/2} q_i(\omega/2)\hat{\phi}(\omega/2)$, $\omega \in \mathbb{R}^n$, for some wavelet masks q_i, $1 \leq i \leq r$, and a *refinable function* $\phi \in L_2(\mathbb{R}^n)$ that satisfies $\hat{\phi}(\omega) = 2^{-n/2}\tau(\omega/2)\hat{\phi}(\omega/2)$, $\omega \in \mathbb{R}^n$, for some lowpass mask τ. We denote by $\hat{g}$ the Fourier transform of the $L_2(\mathbb{R}^n)$ function g, i.e. for $g \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, $\hat{g}(\omega) = \int_{\mathbb{R}^n} g(x)e^{-ix\cdot\omega}\mathrm{d}x$, for $\omega \in \mathbb{R}^n$. For each $1 \leq i \leq r$ and $l \in \mathbb{Z}, k \in \mathbb{Z}^n$, we define $\psi^{(i)}_{l,k} := 2^{ln/2}\psi^{(i)}(2^l \cdot -k)$, which is a scaled, translated version of the mother wavelet $\psi^{(i)}$.

If Λ is a frame, i.e., if it satisfies, for some $A, B > 0$:

$$A\|g\|^2 \leq \sum_{\psi \in \Lambda} |\langle g, \psi\rangle|^2 \leq B\|g\|^2$$

for all $g \in L_2(\mathbb{R}^n)$, we say that Λ is a *(MRA-based) wavelet frame*. In the case that $A = B$, Λ is a *tight frame*, and we call it a *(MRA-based) tight wavelet frame*.

It is well known that for a (MRA-based) tight wavelet frame $\Lambda(\psi^{(1)}, \ldots, \psi^{(r)})$, the number r of mother wavelets is necessarily at least $2^n - 1$, and the number is minimal (i.e. $r = 2^n - 1$) when it is an orthonormal wavelet basis. In a (MRA-based) orthonormal wavelet system, the accuracy number of the lowpass mask τ determines the number of vanishing moments of the associated highpass masks. In any wavelet system, if all of the highpass masks have $l \geq 1$ vanishing moments, all polynomials of degree at most l lie in the subspace of translations of the refinable function. As l increases, this results in faster convergence in the approximation of L_2 functions by the wavelet system [28].

The next result is the *unitary extension principle* (UEP), which provides a systematic way to construct a tight wavelet frame [16, 25]. It consists of a set of conditions on a collection of trigonometric polynomials τ, q_i, where $1 \leq i \leq r$, such that $\Lambda(\psi^{(1)}, \ldots, \psi^{(r)})$ (see Equation (1)) is a tight frame. The following version of the theorem comes from [17]:

Result 1 (UEP). *Let τ be a trigonometric polynomial with $\tau(0) = 2^{n/2}$, and let ϕ be defined by $\hat{\phi}(\omega) := \prod_{j=0}^{\infty} 2^{-n/2}\tau(2^{-j}\omega)$ for $\omega \in \mathbb{R}^n$. If q_i, $1 \le i \le r$, are trigonometric polynomials such that for all $\omega \in \mathbb{T}^n$ and $\gamma \in \{0, \pi\}^n$:*

$$\tau(\omega)\overline{\tau(\omega+\gamma)} + \sum_{i=1}^{r} q_i(\omega)\overline{q_i(\omega+\gamma)} = \begin{cases} 2^n, & \gamma = 0 \\ 0, & \textit{otherwise}, \end{cases}$$

then $\Lambda(\psi^{(1)}, \ldots, \psi^{(r)})$ is a tight wavelet frame in $L_2(\mathbb{R}^n)$.

In this paper, when a set of wavelet masks q_i, $i = 1, \ldots, r$ satisfy the UEP conditions as above with some lowpass mask τ, we will call them an *extensible set of (wavelet) masks for τ*, or simply an *extensible set*, if the lowpass mask is clear.

2.3. Sos Representations and Construction of Extensible Sets

In [4] and [22], the authors make use of the UEP conditions in Result 1 to describe methods by which a multidimensional lowpass filter satisfying certain conditions may be used to create a multidimensional tight wavelet frame. These make use of the idea of finding a sum of hermitian squares (sos) representation of a nonnegative trigonometric polynomial; i.e., given a nonnegative trigonometric polynomial $g : \mathbb{T}^n \to \mathbb{R}$, we say that g has a sos representation, or that g is a sos, if there exist trigonometric polynomials g_j, $1 \le j \le M < +\infty$ such that

$$g(\omega) = \sum_{j=1}^{M} |g_j(\omega)|^2, \quad \forall \omega \in \mathbb{T}^n. \tag{2}$$

If τ is a multidimensional refinement mask satisfying the sub-QMF condition, which is to say that

$$f(\tau;\omega) := 1 - 2^{-n} \sum_{\gamma \in \{0,\pi\}^n} |\tau(\omega/2+\gamma)|^2 \ge 0, \quad \forall \omega \in \mathbb{T}^n \tag{3}$$

and if $f(\tau;\omega)$ is a sum of hermitian squares, then the functions appearing in a sos representation of $f(\tau;\omega)$ may be used to obtain explicitly an extensible set for the refinement mask τ. For a general multidimensional refinement mask satisfying the sub-QMF condition, however, it is difficult to determine whether the associated function $f(\tau;\omega)$ is a sos, and even if it is known that $f(\tau;\omega)$ is a sos, finding a sos representation for $f(\tau;\omega)$ may not be straightforward. When context makes it clear what the lowpass mask associated with $f(\tau;\cdot)$ is, we will usually shorten this to $f(\cdot)$ or just f.

We let $\Gamma := \{0,1\}^n$, which we think of as a particular set[4] of distinct coset representatives of the set $\mathbb{Z}^n/2\mathbb{Z}^n$, let $\Gamma' := \Gamma \setminus \{0\}$, and recall the *polyphase*

[4]Throughout this work, we may choose Γ as any set of distinct coset representatives of $\mathbb{Z}^n/2\mathbb{Z}^n$ containing 0 (so long as this choice is consistent throughout). The particular choice for Γ is made in the current paper only to make our presentation more concise.

representation of a mask. For $n \geq 1$, and given a mask τ with corresponding filter h, the *polyphase component* of τ associated with the coset $\nu \in \Gamma$ is

$$\tau_\nu(\omega) := 2^{-n/2} \sum_{k \in \mathbb{Z}^n} h(2k - \nu) e^{-ik\cdot\omega}. \tag{4}$$

Thus $\tau(\omega) = \sum_{\nu\in\Gamma} \tau_\nu(2\omega) e^{i\nu\cdot\omega}$. Using $\sum_{\gamma\in\{0,\pi\}^n} |\tau(\omega+\gamma)|^2 = 2^n \sum_{\nu\in\Gamma} |\tau_\nu(2\omega)|^2$, the function $f(\tau;\omega)$ in (3) can be written in terms of the polyphase components of τ as well:

$$f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(\omega)|^2. \tag{5}$$

The following theorem is taken from [22], but has been adapted to the notation of this paper. It provides the connection between sos representations of $f(\tau;\cdot)$ and construction of an extensible set for the lowpass mask τ.

Result 2 (Theorem 3.4 of [22])**.** *Suppose τ is a lowpass mask that satisfies the sub-QMF condition, and $\sum_{\nu\in\Gamma} |\tau_\nu(\omega)|^2 + \sum_{j=1}^{M} |g_j(\omega)|^2 = 1$, for all $\omega \in \mathbb{T}^n$. Then the $2^n + M$ functions*

$$q_{1,\mu}(\omega) = \sum_{\nu\in\Gamma} e^{i\nu\cdot\omega}(\delta_{\nu,\mu} - \tau_\nu(2\omega)\overline{\tau_\mu(2\omega)}), \quad \mu \in \Gamma,$$

$$q_{2,j}(\omega) = -\tau(\omega)\overline{g_j(2\omega)}, \quad j = 1, \ldots, M,$$

satisfy the UEP conditions with τ, and thus form an extensible set for τ.

The related, stronger condition called the QMF condition (where $f(\tau;\cdot) = 0$) is studied extensively in the wavelet literature [27, 29], especially in the context of orthonormal wavelet basis construction for the univariate case [1, 30].

While there are several tight wavelet frame constructions in the univariate case (c.f. [7] and references within), multidimensional tight wavelet frame constructions have mostly been done in rather limited settings. For example, many constructions work only for two dimensions (e.g. [15, 19, 24]), some shed little insight on how to find explicit lowpass filters satisfying their conditions (e.g. [4]), and others construct families of multidimensional tight wavelet frames for very specific theoretical goals (e.g. [14, 17]). Some notable exceptions to this trend include [5, 18], in which tight wavelet frames are constructed for box splines in any dimension, and [3], which considers the case of nonnegative lowpass filter coefficients, though all three of these papers have many contributions beyond those briefly mentioned here. For many additional references about multidimensional tight wavelet frame construction, see [19].

3. Coset Sum Tight Wavelet Frames: Theory and Constructions

In Section 3.1, we begin with the definition of the coset sum operator and compute the function $f(\tau;\cdot)$ when τ is a lowpass mask obtained from the coset sum operator. In Section 3.2, we describe a general and basic way of constructing a Hermitian matrix P and vector of complex exponential functions x such that $g = x^*Px$, for some nonnegative trigonometric polynomial g, and under some conditions on this polynomial, we show that we can modify P in a way that makes it positive semidefinite, which leads to a simple method for finding sos representations of $f(\tau;\cdot)$. In Section 3.3, we consider interpolatory input masks to the coset sum, which result in interpolatory multidimensional lowpass masks τ with a particular structure that we exploit to construct a sos representation for $f(\tau;\cdot)$ using the Fejér-Riesz Lemma.

In Section 3.4, we present our general method for constructing sos representations for $f(\tau;\cdot)$, for τ the output of the coset sum method in n dimensions, for some $n \geq 2$. Here, we construct a positive semidefinite matrix P and vector of complex exponential functions x such that $f = x^*Px$ as before, but the structure of the coset sum operator is used to decrease the size of this matrix significantly from that of Section 3.2. After this, some discussion of the conditions for this construction method to succeed, and some stronger sufficient conditions are presented.

3.1. $f(\tau;\cdot)$ for Coset Sum Generated Lowpass Masks τ

Let us recall the definition of the coset sum operator [21]. Let $R(\omega) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} H(k)e^{-ik\omega}$, for $\omega \in \mathbb{T}$, be the Fourier transform of the univariate lowpass filter H, so that $R(0) = \sqrt{2}$. Then the output of the coset sum operator into n dimensions is given by

$$\tau(\omega) := \mathcal{C}_n[R](\omega) := \frac{1}{2^{n/2}}\left(2 - 2^n + \sqrt{2}\sum_{\nu\in\Gamma'} R(\nu\cdot\omega)\right), \quad \omega \in \mathbb{T}^n$$

where $\Gamma' = \Gamma \setminus \{0\} = \{0,1\}^n \setminus \{0\}$, and the associated filter h is defined via $\tau(\omega) =: 2^{-n/2}\sum_{k\in\mathbb{Z}^n} h(k)e^{-ik\cdot\omega}$. In general, coset sum generated lowpass filters in n dimensions have a star shape, as can be seen in the case of two dimensions in Figures 1(a) and 2, as well as [21, Figures 4,5].

We find the polyphase components of τ below. For $\nu = 0$, from the definition in Equation (4), and the fact that $h(k) \neq 0$ only if $k \in \bigcup_{\nu\in\Gamma'} \operatorname{span}(\nu)$, we have

$$\tau_0(\omega) = \frac{1}{2^{n/2}}\left(2 - 2^n + (2^n - 1)H(0) + \sum_{\nu\in\Gamma'}\sum_{k\neq 0} H(2k)e^{-ik\nu\cdot\omega}\right).$$

For $\nu \in \Gamma'$, we use again the fact that $h(k) \neq 0$ only when $k \in \bigcup_{\nu\in\Gamma'} \operatorname{span}(\nu)$, and note that $2k - \nu = c\tilde{\nu}$ implies $-c\tilde{\nu} \equiv \nu (\operatorname{mod} 2\mathbb{Z}^n)$, and for $\tilde{\nu} \in \Gamma$, if $c \in 2\mathbb{Z}$, then $c\tilde{\nu} \in 2\mathbb{Z}^n$, and otherwise $c\tilde{\nu} \equiv \tilde{\nu} (\operatorname{mod} 2\mathbb{Z}^n)$. Since Γ contains distinct coset representatives, $\tilde{\nu} = \nu$ and thus $k \in \operatorname{span}(\nu)$. Then $k = (c+1)\nu/2$, and letting $c = 2j - 1$:

$$\sum_{k\in\mathbb{Z}^n} h(2k-\nu)e^{-ik\cdot\omega} = \sum_{j\in\mathbb{Z}} h((2j-1)\nu)e^{-ij\nu\cdot\omega} = \sum_{j\in\mathbb{Z}} H(2j-1)e^{-ij\nu\cdot\omega},$$

hence we have, for $\nu \in \Gamma'$,

$$\tau_\nu(\omega) = \frac{1}{2^{n/2}} \sum_{j \in \mathbb{Z}} H(2j-1) e^{-ij\nu\cdot\omega}. \tag{6}$$

Now we are ready to write $f(\tau;\cdot)$ for the coset sum generated lowpass filter τ. Using the observations in (5) and above, we have that

$$\begin{aligned} f(\tau;\omega) = 1 &- \frac{(2-2^n+(2^n-1)H(0))^2}{2^n} \\ &- \frac{2-2^n+(2^n-1)H(0)}{2^{n-1}} \sum_{\nu\in\Gamma'} \sum_{k\in\mathbb{Z}\setminus\{0\}} H(2k)\cos(k\nu\cdot\omega) \\ &- \frac{1}{2^n} \sum_{\nu,\gamma\in\Gamma'} \sum_{j,k\in\mathbb{Z}\setminus\{0\}} H(2k)H(2j)e^{-i(k\nu-j\gamma)\cdot\omega} \\ &- \frac{1}{2^n} \sum_{\nu\in\Gamma'} \sum_{j,k\in\mathbb{Z}} H(2k-1)H(2j-1)e^{-i(k-j)\nu\cdot\omega}. \end{aligned}$$

Simplifying, we may write $f(\tau;\omega)$ as:

$$\alpha - \sum_{\nu\in\Gamma'} \sum_{k\geq 1} \alpha(k)\cos(k\nu\cdot\omega) - \frac{2}{2^n} \sum_{\substack{\nu,\gamma\in\Gamma' \\ \nu >_{\mathrm{lex}} \gamma}} \sum_{j,k\in\mathbb{Z}\setminus\{0\}} H(2k)H(2j)\cos((k\nu-j\gamma)\cdot\omega),$$

where

$$\alpha = 1 - \frac{(2-2^n+(2^n-1)H(0))^2}{2^n} - \frac{2^n-1}{2^n} \sum_{j\neq 0} H(j)^2, \tag{7}$$

and for $k \geq 1$:

$$2^{n-1}\alpha(k) = (2^n-2)(H(0)-1)(H(2k)+H(-2k)) + \sum_{j\in\mathbb{Z}} H(j)H(j+2k). \tag{8}$$

Here and below, we use $\geq_{\mathrm{lex}}$ to denote the lexicographical order on $\mathbb{Z}^n$. That is, for $x, y \in \mathbb{Z}^n$, $x \geq_{\mathrm{lex}} y$ if $x = y$, or if $x \neq y$, and in the first position (reading left to right) such that $x_i \neq y_i$, $x_i > y_i$. We will write $>_{\mathrm{lex}}$ to denote the case when equality is excluded. The choice of order here is unimportant (under some mild assumptions), though we use lexicographic for its convenience.

3.2. Sos Representations from Matrix Factorizations: Naive Construction

In this subsection we first make some observations about sos representations from positive semidefinite matrices for general nonnegative trigonometric polynomials, and then apply them for the special case when the polynomial is $f(\tau;\cdot)$. We will use the notation $M_r(k)$ to denote the space of $r \times r$ matrices with entries in the field k, for a positive integer r.

We observe that a nonnegative trigonometric polynomial g has a sos representation if and only if there exists a positive semidefinite matrix P and a vector $x = [e^{-ik\cdot\omega}]_{k\in\mathcal{I}}$, for some finite set $\mathcal{I} \subseteq \mathbb{Z}^n$, such that $g = x^*Px$. Indeed, given such a P, the Cholesky factorization of $P = LL^*$ for a lower triangular matrix L (or, indeed any representation of the form $P = AA^*$ for a matrix A), gives a sos representation of g with generators L^*x (each entry of which is seen to be a trigonometric polynomial), since $g = x^*Px = (L^*x)^*(L^*x)$. Conversely, given a sos representation of g as in Equation (2), if for each j, $g_j(\omega) = \sum_{k\in\mathbb{Z}^n} c_{j,k}e^{-ik\cdot\omega}$, and we let $\mathcal{I} = \bigcup_{j=1}^{M}\{k \in \mathbb{Z}^n : c_{j,k} \neq 0\} \cup \{0\}$ with some ordering, then we may form the matrix A of size $|\mathcal{I}| \times M$, with $A_{k,j} = \overline{c_{j,k}}$ for $1 \leq j \leq M$, $k \in \mathcal{I}$, and this gives the representation $g = x^*(AA^*)x$ for this A and $x = [e^{-ik\cdot\omega}]_{k\in\mathcal{I}}$, where clearly AA^* is positive semidefinite.

Remark 1. Let g be a nonnegative trigonometric polynomial, such that $g(\omega) = \sum_{k\in\mathbb{Z}^n} c_k e^{-ik\cdot\omega}$, with real coefficients c_k, $k \in \mathbb{Z}^n$. Let $x = [e^{-ik\cdot\omega}]$, for some ordering of the set $\{k \in \mathbb{Z}^n : k \geq_{\text{lex}} 0, c_k \neq 0\} \cup \{0\}$ with 0 as the last entry. Observe that because g is real-valued and has real coefficients, $c_k = c_{-k}$. Consider the matrix P_1, with nonzero entries only in its last row and column, and indexed in the same way as x, such that $(P_1)_{0,0} = c_0$, and $(P_1)_{0,k} = (P_1)_{k,0} = c_k$. Clearly, we have $g = x^*P_1x$, but typically, P_1 will not be positive semidefinite. To see this, suppose for example that c_0 and c_k are both nonzero for some $k >_{\text{lex}} 0$, and $c_0 > 0$. Let y be the vector which has 0 in every entry except at the entries indexed by k and 0, where it is equal to $-1/c_k$ and $1/c_0$, respectively. Then $y^*P_1y = -1/c_0 < 0$.

To remedy this situation, consider P_2, which has the same last row and column as P_1 except $(P_2)_{0,0}$, but $(P_2)_{0,0} = c_0 - \sum_{k>_{\text{lex}}0} |c_k|$, $(P_2)_{k,k} = |c_k|$ for $k >_{\text{lex}} 0$, and $(P_2)_{j,k} = 0$ elsewhere. Then it is clear that P_2 again satisfies $g = x^*P_2x$ for the vector x above, and is weakly diagonally dominant with nonnegative diagonal entries in (at least) all but the last row. If it happens that $(P_2)_{0,0} \geq \sum_{k>_{\text{lex}}0} |c_k|$, then this holds for the last row as well, which implies that P_2 is positive semidefinite. Put differently, if it happens that $c_0 \geq \sum_{k\neq 0} |c_k|$, the matrix P constructed in this way will be weakly diagonally dominant and positive semidefinite. $\square$

The following simple lemma formalizes the idea from the remark above, namely redistributing the constant term of the trigonometric polynomial g along the diagonal in an effort to make the matrix P positive semidefinite, as in the change from P_1 to P_2. A more general version of this idea is found in [23].

Lemma 1. [Change of Diagonal] *Let g be a nonnegative trigonometric polynomial, such that $g(\omega) = \sum_{k\in\mathbb{Z}^n} c_k e^{-ik\cdot\omega}$, with real coefficients c_k. Let $J = \{k \in \mathbb{Z}^n : c_k \neq 0\} \cup \{0\}$ with some ordering, and for some nonempty $S \subseteq J$ with the inherited ordering, let $x = [e^{-ik\cdot\omega}]_{k\in S}$, and $P \in M_{|S|}(\mathbb{R})$ be a Hermitian matrix such that $g = x^*Px$. Suppose that a diagonal matrix $D \in M_{|S|}(\mathbb{R})$ satisfies $\sum_{i\in S} D_{i,i} = 0$ and $P_{i,i} + D_{i,i} \geq \sum_{j\in S, j\neq i} |P_{i,j}|$ for all $i \in S$. Then $P + D$ is positive semidefinite (and weakly diagonally dominant), and g has a sos representation.*

By choosing the trigonometric polynomial g in Remark 1 as $f(\tau;\cdot)$, with τ the lowpass mask output by the coset sum method that satisfies the sub-QMF condition $f(\tau;\cdot) \geq 0$, after applying Result 2, we obtain the *naive construction method* for extensible sets with coset sum lowpass masks. In Section 3.4, we introduce a more sophisticated method than this one for generating a matrix P and vector x satisfying $f(\tau;\cdot) = x^* P x$. Without assuming any special structure for the input mask to the coset sum, the method described in Section 3.4 will typically result in a significantly smaller matrix than the one described here, though the sos generators are likely to be more complicated. Depending on the preferences of the filter designer, then, it may be beneficial to compare these approaches to obtaining the sos representation of f and the resulting frames. The naive method described here will typically result in many more sos generators (and thus wavelet masks), which have a simple form if the Cholesky factorization is used. The method of Theorem 2 in Section 3.4 will typically result in far fewer sos generators, but these may be more complicated.

3.3. Sos Representations from the Fejér-Riesz Lemma: Interpolatory Input Masks

It is easy to see that the coset sum operator $\mathcal{C}_n$ preserves the positive accuracy and the interpolatory properties (cf. [21]). Our result below shows that when the coset sum lowpass mask τ, or equivalently the input univariate mask R, is interpolatory, the univariate sub-QMF condition on R is sufficient to give a sos representation for $f(\tau;\cdot)$, hence an extensible set for τ. In obtaining this result, we use the Fejér-Riesz Lemma, the statement and proof of which may be found in [6]. We now present our result for interpolatory inputs to the coset sum method.

Theorem 1. *Let R be a univariate, positive accuracy, interpolatory mask with corresponding filter H, such that R satisfies the sub-QMF condition, and $R(0) = \sqrt{2}$. Let τ be the output of the coset sum into n dimensions, for some $n \geq 2$. Then $f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(\omega)|^2$ has a sos representation, and there is an extensible set for τ with $2^{n+1} - 1$ highpass filters.*

Proof. By the interpolatory condition, $R_0(\omega) = 2^{-1/2}$, and we have that

$$f(R;\omega) = \frac{1}{2}\left(1 - \left|\sum_k H(2k-1)e^{-ik\omega}\right|^2\right) \geq 0, \quad \text{for } \omega \in \mathbb{T}.$$

Then if τ is the output of the coset sum into n dimensions, τ must also be interpolatory and have positive accuracy, and since $\tau_0(\omega) = 2^{-n/2}$, we have from Equation (6):

$$\begin{aligned} f(\tau;\omega) &= 1 - 2^{-n} - 2^{-n} \sum_{\nu\in\Gamma'} \left|\sum_k H(2k-1)e^{-ik\nu\cdot\omega}\right|^2 \\ &= \frac{1}{2^n}\sum_{\nu\in\Gamma'}\left(1 - \left|\sum_k H(2k-1)e^{-ik\nu\cdot\omega}\right|^2\right) \\ &= \frac{2}{2^n}\sum_{\nu\in\Gamma'} f(R;\nu\cdot\omega), \quad \text{for } \omega \in \mathbb{T}^n. \end{aligned} \tag{9}$$

Since we have $f(R;\omega) \geq 0$ for $\omega \in \mathbb{T}$ from above, by the Fejér-Riesz Lemma, $f(R;\omega) = |p(\omega)|^2$, $\omega \in \mathbb{T}$, for some trigonometric polynomial p, so we have that

$$f(\tau;\omega) = \sum_{\nu\in\Gamma'} \left| \sqrt{\frac{2}{2^n}} p(\nu\cdot\omega) \right|^2, \quad \text{for } \omega \in \mathbb{T}^n,$$

which is a sos representation of $f(\tau;\omega)$ with $2^n - 1$ sos generators. That there exists an extensible set with τ as the lowpass mask is then the content of Result 2. □

3.4. Sos Representations from Matrix Factorizations: General Input Masks

In the theorem below, we provide a condition (i.e. Condition ($\diamondsuit$)) on the univariate mask R for the existence of a positive semidefinite matrix P and a vector of complex exponentials x such that $f(\tau;\cdot) = x^*Px$ when $\tau = \mathcal{C}_n[R]$ is the coset sum lowpass mask generated from R, which in turn implies that $f(\tau;\cdot)$ has a sos representation. See Remark 4 for a discussion of the relationships between Condition ($\diamondsuit$), the sub-QMF condition for τ, the sub-QMF condition for R, and the existence of a sos representation for $f(\tau;\cdot)$.

Theorem 2. *Let R be a positive accuracy mask with lowpass filter H, such that $R(0) = \sqrt{2}$, and let n be an integer at least 2. Let τ be the output of the coset sum method into n dimensions with input R, and let $f(\tau;\omega) = 1 - \sum_{\nu\in\Gamma} |\tau_\nu(\omega)|^2$. If H satisfies the following condition:*

$$\alpha(k) \geq 0 \textit{ for all } k \geq 1, \textit{ and } H(2k)H(2j) \geq 0 \textit{ for all } k, j \in \mathbb{Z}\setminus\{0\}, \qquad (\diamondsuit)$$

where $\alpha(k)$ is defined as in (8), then $f = x^*Px$ for a vector of complex exponentials x, where P is positive semidefinite (and weakly diagonally dominant), and thus an extensible set exists for τ.

Proof. We begin by constructing a Hermitian matrix Q and a vector of complex exponentials x such that $x^*Qx = f$. Let

$$N = \min\{2l : l \in \mathbb{Z}, l \geq 0, H(k) = H(-k) = 0 \text{ for all } k > 2l\}. \qquad (10)$$

Let $J = \{(0,0)\} \cup \{(\nu,k) : \nu \in \Gamma', k \in \{-N/2,\dots,N/2\} \setminus \{0\}\}$, ordered in blocks $(\nu,-N/2),\dots,(\nu,-1),(\nu,1),\dots,(\nu,N/2)$, for $\nu \in \Gamma'$ in some ordering, with $(0,0)$ as the last element. Let $x = [e^{-ik\nu\cdot\omega}]_{(\nu,k)\in J}$, and for $(\gamma,j),(\nu,k) \in J$:

$$Q_{(\gamma,j),(\nu,k)} = \begin{cases} \alpha & \nu=\gamma=0 \\ -\alpha(k)/2 & \nu\neq 0, \gamma=0, k>0 \\ -\alpha(j)/2 & \gamma\neq 0, \nu=0, j>0 \\ -2^{-n}H(2k)H(2j) & \nu,\gamma\in\Gamma', \nu\neq\gamma \\ -\alpha(N/2-j)/2 & \nu=\gamma\in\Gamma', k=N/2, j<0 \\ -\alpha(N/2-k)/2 & \nu=\gamma\in\Gamma', j=N/2, k<0 \\ 0 & \text{otherwise,} \end{cases}$$

where α and $\alpha(k)$ are defined as in (7) and (8), respectively. By inspection of the product, we see that $f = x^*Qx$. We now apply Lemma 1 to Q to obtain the matrix $P = Q + D$, where as in Remark 1, we add the sum of the magnitudes of the off-diagonal entries to each of the diagonal entries in all but the last row, and subtract the sum of these new diagonal entries from the last diagonal entry. That is, we let D be the diagonal matrix with, for $(\nu, k) \in J$, $D_{(\nu,k),(\nu,k)} = \beta(k)$ for $\nu \neq 0$, and $D_{(0,0),(0,0)} = -(2^n - 1)\sum_{s=-N/2, s\neq 0}^{N/2} \beta(s)$, where

$$\beta(k) = \begin{cases} \frac{2^n-2}{2^n}|H(2k)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + |\alpha(N/2-k)|/2 & \text{if } k < 0, \\ \frac{2^n-2}{2^n}|H(2k)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + |\alpha(k)|/2 & \text{if } 0 < k < N/2, \\ \frac{2^n-2}{2^n}|H(N)|\sum_{j=-N/2,j\neq 0}^{N/2}|H(2j)| + \sum_{j=N/2}^{N}|\alpha(j)|/2 & \text{if } k = N/2. \end{cases}$$

Let $P = Q + D$. Then by Lemma 1, P also satisfies $x^*Px = f$, and it remains to check that Condition ($\diamondsuit$) implies the positive semidefiniteness and weak diagonal dominance of P.

We see that for $\nu \neq 0$, $P_{(\nu,k),(\nu,k)} = D_{(\nu,k),(\nu,k)} = \sum_{(\gamma,j)\neq(\nu,k)} |Q_{(\nu,k),(\gamma,j)}|$. The equality $P_{(\nu,k),(\nu,k)} = \sum_{(\gamma,j)\neq(\nu,k)} |Q_{(\nu,k),(\gamma,j)}|$ holds when $\nu = 0$ as well if we have

$$\alpha - (2^n - 1)\sum_{\substack{k=-N/2 \\ k\neq 0}}^{N/2} \beta(k) = \frac{2^n - 1}{2}\sum_{k=1}^{N/2}|\alpha(k)|.$$

By Condition ($\diamondsuit$), this is equivalent to:

$$\alpha - \frac{(2^n-1)(2^n-2)}{2^n}\sum_{\substack{j,k=-N/2 \\ j,k\neq 0}}^{N/2} H(2k)H(2j) - (2^n - 1)\sum_{k=1}^{N}\alpha(k) = 0,$$

the left hand side of which is just $f(0)$, which equals 0 by the positive accuracy condition. Thus, we can apply the last part of Lemma 1 to say that P is positive semidefinite, and f has a sos representation. That an extensible set exists with τ as the lowpass mask is then the content of Result 2. □

Remark 2. The matrix P in the proof clearly has a block matrix structure. More precisely, we define a vector $v \in \mathbb{R}^N$ of length N as in Equation (10), and Hermitian matrices $B, C \in M_N(\mathbb{R})$ of order N as

$$v = [0, \cdots, 0, -\alpha(1)/2, \cdots, -\alpha(N/2)/2]$$

$$B = \begin{bmatrix} \beta(-N/2) & & & & & -\alpha(N)/2 \\ & \ddots & & & & \vdots \\ & & \beta(-1) & & & -\alpha(N/2+1)/2 \\ & & & \beta(1) & & \\ & & & & \ddots & \\ -\alpha(N)/2 & \cdots & -\alpha(N/2+1)/2 & & & \beta(N/2) \end{bmatrix}$$

$$C = -\frac{1}{2^n}\begin{bmatrix} H(-N)^2 & H(-N)H(2-N) & \cdots & H(-N)H(N) \\ H(2-N)H(-N) & H(2-N)^2 & \cdots & H(2-N)H(N) \\ \vdots & \vdots & \ddots & \vdots \\ H(N)H(-N) & H(N)H(2-N) & \cdots & H(N)^2 \end{bmatrix},$$

where H is the univariate lowpass filter, and $\alpha(k)$ and $\beta(k)$ are the parameters determined by H as in the proof. Note that for the matrix C the zero index is absent, so in the first row (or column) we have $H(-N)H(-2)$ followed by $H(-N)H(2)$. Then the matrix P is given as

$$P = \begin{bmatrix} B & C & C & \cdots & C & v^T \\ C & B & C & \cdots & C & v^T \\ C & C & B & \ddots & C & v^T \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ C & C & C & \cdots & B & v^T \\ v & v & v & \cdots & v & b \end{bmatrix}, \tag{11}$$

where under Condition ($\diamondsuit$), $b = \frac{2^n-1}{2}\sum_{k=1}^{N/2}|\alpha(k)|$. $\square$

Remark 3. In the above proof and Remark 2, we chose N to be even, since this makes the indexing of the matrix P simpler, but for some filters, this choice essentially corresponds to zero-padding the outside of the filter to extend the support. The effect of this on the matrix P is that zero rows and columns may appear for certain filter inputs, and these indices can be removed from the index set J (and the corresponding rows and columns from P and x) with no change to the equality $f = x^*Px$, since the zero rows and columns of P do not contribute to this product. In our examples, we will always present P with any zero rows and columns removed. $\square$

Remark 4. There are several conditions on the univariate mask R at play in the surrounding discussion. Let $n \geq 2$, and let τ be the output of the coset sum with input R in n dimensions. Consider the following statements:

(i) R is interpolatory (or equivalently, τ is interpolatory)
(ii) R satisfies the univariate sub-QMF condition,
(iii) $f(R;\cdot)$ has a sos representation,
(iv) τ satisfies the sub-QMF condition,
(v) $f(\tau;\cdot)$ has a sos representation,
(vi) Condition ($\diamondsuit$) holds.

By the Fejér-Riesz Lemma, (ii) and (iii) are equivalent. (vi) implies (v) by Theorem 2, and (v) clearly implies (iv). Under (i), (ii)/(iii) implies (v) by Theorem 1, and (iv) implies (ii)/(iii) by Equation (9), so (ii), (iii), (iv), and (v) are all equivalent in this case. (vi) is strictly stronger than (ii)/(iii), even under (i), as seen in Example 2 in the next section. If $n = 2$, then (iv) and (v) are equivalent by Theorem 2.4 of [4]. For $n \geq 3$, it is unknown if (iv) implies (v), but by Theorem 2.5 of [4], (iv) does not imply (v) if τ is a general multidimensional lowpass mask (i.e., not coming from the coset sum). $\square$

The following result provides a partial converse to Theorem 2.

Proposition 1. *Let R, H, τ, n, and f be as in Theorem 2, with H not necessarily satisfying Condition ($\diamondsuit$), and let x and P be as in its proof so that $f = x^*Px$. If P is weakly diagonally dominant (hence positive semidefinite), then H satisfies Condition ($\diamondsuit$).*

Proof. Suppose $Q \in M_K(\mathbb{R})$ is a square, Hermitian, weakly diagonally dominant matrix with nonnegative diagonal entries, for K some positive integer, such that $e^*Qe = 0$, for e the column vector of all ones with length K. Then $0 = \sum_{i,j=1}^{K} Q_{i,j} \geq \sum_{i=1}^{K}(Q_{i,i} - \sum_{j\neq i}|Q_{i,j}|) \geq 0$, by the weak diagonal dominance of Q. Moreover, each of the summands $Q_{i,i} - \sum_{j\neq i}|Q_{i,j}| \geq 0$, so this equality forces $Q_{i,i} = \sum_{j\neq i}|Q_{i,j}|$ for all $1 \leq i \leq K$. Since the first inequality must be an equality, rearranging gives $\sum_{i=1}^{K}\sum_{j\neq i}(Q_{i,j} + |Q_{i,j}|) = 0$, and since each of the summands is nonnegative, $Q_{i,j} = -|Q_{i,j}|$ for all $1 \leq i, j \leq K$, $i \neq j$.

In the present case, the equality $f = x^*Px$ is clear by inspection of this product. Since $0 = f(0) = e^*Pe$ from the positive accuracy condition, the conditions on P imply that we may apply the above result to P, which gives us Condition ($\diamondsuit$). $\square$

The next corollary of Theorem 2 shows some simple sufficient conditions for Condition ($\diamondsuit$) to hold, hence for the associated lowpass mask to give rise to an extensible set. The fact that these conditions are not necessary can be seen easily, for example, by observing that many filters in Example 4 in the next section do not satisfy the conditions in the corollary but satisfy Condition ($\diamondsuit$). It should be noted that under these conditions, the filter coefficients $h(k)$ of τ are nonnegative for all $k \in \mathbb{Z}^n$, a case which has also been studied in [3] without the coset sum structure on the lowpass filter.

Corollary 1. *Let R be a positive accuracy mask with lowpass filter H, such that $R(0) = \sqrt{2}$. Suppose that $H(k) \geq 0$ for all $k \in \mathbb{Z}$, and $H(0) \geq (2^n-2)/(2^n-1)$, for some integer $n \geq 2$. Then H satisfies Condition ($\diamondsuit$) for this n, and thus an extensible set exists with $\tau = \mathcal{C}_n[R]$ as the lowpass mask.*

Proof. By Theorem 2, it suffices to show that the filter H satisfies Condition ($\diamondsuit$). Clearly, $H(2k)H(2j) \geq 0$ for all $j, k \neq 0$, and it remains to check that $\alpha(k) \geq 0$ for each k. We observe that

$$2^{n-1}\alpha(k) = ((2^n-1)H(0)-(2^n-2))(H(2k)+H(-2k)) + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2k,0}} H(j)H(j+2k),$$

and clearly the last sum and $H(2k) + H(-2k)$ are nonnegative, so $\alpha(k) \geq 0$ if $(2^n - 1)H(0) \geq 2^n - 2$, which is the stated condition on $H(0)$. $\square$

4. Examples

We illustrate our findings in the previous section by presenting some examples in this section. The first example is the extensible set constructed from the coset sum lowpass filter with the input B-spline of order 2.

Example 1. [From Interpolatory B-spline Filter of Order 2] For the (centered) B-spline of order 2 (also called the centered hat function), we consider the interpolatory mask $R(\omega) = 2^{-1/2}(1 + \cos(\omega))$, $\omega \in \mathbb{T}$. Then, for $n \geq 2$, the n-dimensional coset sum lowpass mask is $\tau(\omega) = 2^{-n/2}(1 + \sum_{\nu \in \Gamma'} \cos(\nu \cdot \omega))$, $\omega \in \mathbb{T}^n$. From Equation (9) in the proof of Theorem 1, we see that

$$f(\tau;\omega) = \frac{2}{2^n} \sum_{\nu \in \Gamma'} f(R;\nu\cdot\omega) = \sum_{\nu \in \Gamma'} \frac{1}{2^{n+1}}(1-\cos(\nu\cdot\omega)) = \sum_{\nu \in \Gamma'} \left| \frac{1}{2^{\frac{n}{2}+1}}(1 - e^{-i\nu\cdot\omega}) \right|^2,$$

which gives us a sos representation for f, where the simple identity $2(1-\cos\omega) = |1-e^{-i\omega}|^2$, $\omega \in \mathbb{T}$ (which may be seen as a simple application of the Fejér-Riesz Lemma), is used for the last equality.

Since the univariate filter H associated with the mask R satisfies Condition ($\diamondsuit$) for all dimensions $n \geq 2$ (in fact, for any interpolatory filter H, Condition ($\diamondsuit$) is independent of n, hence holds true for all $n \geq 2$ if it holds true for any specific n), by Theorem 2, we have a matrix factorization of $f = x^* P x$ with a positive semidefinite matrix $P \in M_{2^n}(\mathbb{R})$ of the form

$$P = \frac{1}{2^{n+2}} \begin{bmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \\ -1 & -1 & \cdots & -1 & 2^n - 1 \end{bmatrix}$$

and $x^* = [(e^{i\nu\cdot\omega})_{\nu\in\Gamma'}, 1]$, where Remark 3 is used for the reduction of the matrix. The Cholesky decomposition of P as $P = LL^*$ is given with the lower triangular matrix

$$L = \frac{1}{2^{1+n/2}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}$$

and in this case, $L^* x$ is exactly the same as the sos representation we obtained above using the approach of Theorem 1.

Either way, we get a sos decomposition of $f(\tau;\omega)$ with $2^n - 1$ generators. Thus, by Result 2, we obtain an extensible set for τ with $2^{n+1}-1$ wavelet masks, which we index by $\mu \in \Gamma$ for the first 2^n, and $\eta \in \Gamma'$ for the last $2^n - 1$:

$$q_{1,\mu}(\omega) = \frac{1}{2^{n+1}} \left[2^{n+1} e^{i\mu\cdot\omega} - (1 + e^{i2\mu\cdot\omega}) \left(1 + \sum_{\nu\in\Gamma'} \cos(\nu\cdot\omega) \right) \right],$$

$$q_{2,\eta}(\omega) = -\frac{1}{2^{n+1}}(1 - e^{i2\eta\cdot\omega}) \left(1 + \sum_{\nu\in\Gamma'} \cos(\nu\cdot\omega) \right).$$

Note that $q_{1,\mu}$ have 2 vanishing moments, but $q_{2,\eta}$ have 1 vanishing moment, and as a result, the tight wavelet frame has 1 vanishing moment.

$$\begin{array}{ccc} & 1/2 & 1/2 \\ 1/2 & \mathbf{1} & 1/2 \\ 1/2 & 1/2 & \end{array}$$

(a) $\tau(\omega)$

$$\begin{array}{ccc} & -1/4 & -1/4 \\ -1/4 & \mathbf{3/2} & -1/4 \\ -1/4 & -1/4 & \end{array}$$

(b) $q_{1,(0,0)}(\omega)$

$$\begin{array}{ccccc} & -1/8 & -1/8 & -1/8 & -1/8 \\ -1/8 & -1/4 & 7/4 & \mathbf{-1/4} & -1/8 \\ -1/8 & -1/8 & -1/8 & -1/8 & \end{array}$$

(c) $q_{1,(1,0)}(\omega)$

$$\begin{array}{ccccc} & 1/8 & 1/8 & -1/8 & -1/8 \\ 1/8 & 1/4 & 0 & \mathbf{-1/4} & -1/8 \\ 1/8 & 1/8 & -1/8 & -1/8 & \end{array}$$

(d) $q_{2,(1,0)}(\omega)$

$$\begin{array}{ccc} & -1/8 & -1/8 \\ -1/8 & \mathbf{-1/4} & -1/8 \\ -1/8 & 7/4 & -1/8 \\ -1/8 & -1/4 & -1/8 \\ -1/8 & -1/8 & \end{array}$$

(e) $q_{1,(0,1)}$

$$\begin{array}{ccc} & -1/8 & -1/8 \\ -1/8 & \mathbf{-1/4} & -1/8 \\ -1/8 & 0 & 1/8 \\ 1/8 & 1/4 & 1/8 \\ 1/8 & 1/8 & \end{array}$$

(f) $q_{2,(0,1)}(\omega)$

$$\begin{array}{ccccc} & & & -1/8 & -1/8 \\ & & -1/8 & \mathbf{-1/4} & -1/8 \\ & -1/8 & 7/4 & -1/8 & \\ -1/8 & -1/4 & -1/8 & & \\ -1/8 & -1/8 & & & \end{array}$$

(g) $q_{1,(1,1)}(\omega)$

$$\begin{array}{ccccc} & & & -1/8 & -1/8 \\ & & -1/8 & \mathbf{-1/4} & -1/8 \\ & 1/8 & 0 & -1/8 & \\ 1/8 & 1/4 & 1/8 & & \\ 1/8 & 1/8 & & & \end{array}$$

(h) $q_{2,(1,1)}(\omega)$

Figure 1: Wavelet filters and lowpass filter from the B-spline of order 2 in Example 1 ($n = 2$).

Figure 1 depicts[5] the filters in dimension 2. In this case we have a sos decomposition of f with 3 generators, and thus 7 wavelet filters. In [18, Example 2.6], another construction is given which also yields a tight frame with 7 wavelet filters, which have smaller support, but decreased directionality and lack of symmetry. In [22, Example 5.2], it is shown that 2 sos generators (hence 6 wavelet filters) are actually sufficient, and they arrive at a very similar filter bank to ours, the main difference being that our $q_{2,(0,1)}$ and $q_{2,(1,1)}$ are essentially combined in their construction to yield one filter with larger support and loss of symmetry. In [5, Example 4.7], the authors construct a tight wavelet frame with this same lowpass filter, but with only 5 wavelet filters, which have smaller support, but decreased directionality and no symmetry. $\square$

Remark 5. It is easy to see that the filters in Figure 1 all have symmetry. In fact, if R is a symmetric filter, i.e., $H(k) = H(-k)$ for all $k \in \mathbb{Z}$, then the output of the coset sum τ will have symmetry through the origin (among other symmetries), so it is a natural question whether or not it is possible to obtain highpass filters with this property. It is not difficult to show that the

[5] In the figures of the support of filters in this paper, the bold-faced number is used to represent the value of the filter at the origin.

highpass filters $q_{1,\mu}$ in Result 2 will have symmetry under this condition, and that the highpass filters $q_{2,j}$ in that result will be symmetric precisely when the sos generators g_j have symmetry. For the sos representations constructed in Theorem 1, this requires $f(R;\omega)$ to have a representation $|p(\omega)|^2$ for symmetric p. For the sos representations constructed in Theorem 2, after constructing P and x as in its proof, this symmetry requires a decomposition of $P = AA^*$ with the property that $A_i^* x$ is symmetric for each $1 \le i \le M$, where A_i is the ith column of the matrix A (see discussion preceding Remark 1). The conditions under which these representations and decompositions exist require further investigation. $\square$

Example 2. [From Interpolatory Deslauriers-Dubuc Filters] For the Deslauriers-Dubuc (DD) filters of order $2k$ [9, 10, 13], $k \ge 1$, we have

$$R(\omega) = \sqrt{2}\cos^{2k}(\omega/2) P_k(\sin^2(\omega/2)),$$

where

$$P_k(x) = \sum_{j=0}^{k-1} \binom{k-1+j}{j} x^j.$$

When $k = 1$, $R(\omega)$ is the B-spline of order 2 mask we discussed already in Example 1. These filters are interpolatory and have positive accuracy, and as proved in [20], these masks satisfy the univariate sub-QMF condition for each $k \ge 1$, and thus by Theorem 1, for each $k \ge 1$ and dimension $n \ge 2$, there is a sos representation for $f(\tau;\cdot)$, where τ is the output of the coset sum with dimension n and input DD mask R of order $2k$.

Since $R_0(\omega) = 2^{-1/2}$ for the DD mask R of order $2k$, we see that

$$f(R;\omega) = 1/2 - \left(R(\omega/2) - 1/\sqrt{2}\right)^2 = R(\omega/2)\left(\sqrt{2} - R(\omega/2)\right),$$

where the fact that $0 \le R(\omega/2) \le \sqrt{2}$ (also proved in [20]) ensures that both factors are nonnegative. Since $f(R;\omega)$ is a univariate nonnegative trigonometric polynomial[6], by the Fejér-Riesz Lemma, there exists a trigonometric polynomial p such that $f(R;\omega) = |p(\omega)|^2$. Since P_k is the unique polynomial of degree $k-1$ satisfying $(1-y)^k P_k(y) + y^k P_k(1-y) = 1$ for all $y \in [0,1]$ (see [6]), we see that $\sqrt{2} - R(\omega/2) = \sqrt{2}(1 - \cos^{2k}(\omega/4) P_k(\sin^2(\omega/4))) = \sqrt{2}\sin^{2k}(\omega/4) P_k(\cos^2(\omega/4))$. Hence $f(R;\omega)$ has a factor of $\sin^{2k}(\omega/4)$, and as a result, $p(\omega)$ has a root of order k at 0. This in turn implies that the sos generators for $f(\tau;\omega)$, with the coset sum generated τ from the DD mask R of order $2k$, have a root of order k at 0, which along with the interpolatory property and the accuracy number of τ implies that all of the wavelet masks in the extensible set constructed from Result 2 have at least k vanishing moments.

[6] Note that $R(\omega/2) - 1/\sqrt{2} = \left(\sum_{k\in\mathbb{Z}} H(2k-1)e^{-i(2k-1)\omega/2}\right)/\sqrt{2}$, so $f(R;\omega) = \frac{1}{2}\left(1 - |\sum_{k\in\mathbb{Z}} H(2k-1)e^{-ik\omega}|^2\right)$, and is therefore properly a trigonometric polynomial in ω, rather than $\omega/2$ as its appearance above might suggest.

For the DD mask of order 4 (i.e. $k = 2$), $R(\omega) = 2^{-1/2}(1 + (9/8)\cos(\omega) - (1/8)\cos(3\omega))$, and for τ the output of the coset sum in dimension $n \geq 2$, we have

$$\begin{aligned} f(\tau;\omega) &= \frac{2^n-1}{2^n}\left(\frac{23}{64}\right) - \frac{2}{2^n}\sum_{\nu\in\Gamma'}\frac{63}{256}\cos(\nu\cdot\omega) - \frac{9}{128}\cos(2\nu\cdot\omega) + \frac{1}{256}\cos(3\nu\cdot\omega) \\ &= \frac{1}{2^{n+7}}\sum_{\nu\in\Gamma'}\left(46 - 63\cos(\nu\cdot\omega) + 18\cos(2\nu\cdot\omega) - \cos(3\nu\cdot\omega)\right). \end{aligned}$$

Thus, if p is a trigonometric polynomial such that

$$2^{n+7}|p(\omega)|^2 = 46 - 63\cos(\omega) + 18\cos(2\omega) - \cos(3\omega),$$

we have $f(\tau;\omega) = \sum_{\nu\in\Gamma'}|p(\nu\cdot\omega)|^2$. One possible such a choice of p is given by

$$2^{(n+7)/2}p(\omega) = \sqrt{\frac{7}{2} - 2\sqrt{3}}(7 + 4\sqrt{3} - e^{-i\omega})(1 - e^{-i\omega})^2.$$

As we can see, here $p(\omega)$ has a double root at $\omega = 0$, which, together with the fact that the mask τ in this case is interpolatory with accuracy number 4, implies that the wavelet masks in the extensible set with lowpass mask τ constructed by Result 2 all have at least 2 vanishing moments. □

Example 3. [From B-spline Filter of Order 3] We see that the (centered) B-spline of order 3, with $R(\omega) = 2^{-5/2}(e^{i\omega} + 3 + 3e^{-i\omega} + e^{-i2\omega})$, satisfies Condition ($\diamondsuit$) only for $n = 2$ and 3. Thus, for example when $n = 2$, by Theorem 2 we have the representation of f as x^*Px with

$$P = \frac{1}{64}\begin{bmatrix} 6 & -1 & -1 & -4 \\ -1 & 6 & -1 & -4 \\ -1 & -1 & 6 & -4 \\ -4 & -4 & -4 & 12 \end{bmatrix}, \quad x = \begin{bmatrix} e^{-i\omega_1} \\ e^{-i\omega_2} \\ e^{-i(\omega_1+\omega_2)} \\ 1 \end{bmatrix}.$$

Finding a Cholesky factorization for P gives $P = LL^*$ with the following L:

$$L = \frac{1}{8}\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \\ -1/\sqrt{6} & \sqrt{35/6} & 0 & 0 \\ -1/\sqrt{6} & -\sqrt{7/30} & 2\sqrt{7/5} & 0 \\ -4/\sqrt{6} & -4\sqrt{7/30} & -2\sqrt{7/5} & 0 \end{bmatrix},$$

which corresponds to a sos representation of f with 3 sos generators:

$$\begin{aligned} 64f(\omega) &= (1/6)|4 + e^{-i(\omega_1+\omega_2)} + e^{-i\omega_2} - 6e^{-i\omega_1}|^2 \\ &\quad + (7/30)|4 + e^{-i(\omega_1+\omega_2)} - 5e^{-i\omega_2}|^2 + (28/5)|1 - e^{-i(\omega_1+\omega_2)}|^2. \end{aligned}$$

Alternatively, using the method described in Remark 1, we obtain the following representation of f as x^*Px:

$$P = \frac{1}{64}\begin{bmatrix} 5 & 0 & 0 & 0 & -5 \\ 0 & 5 & 0 & 0 & -5 \\ 0 & 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ -5 & -5 & -4 & -1 & 15 \end{bmatrix}, \quad x = \begin{bmatrix} e^{-i\omega_1} \\ e^{-i\omega_2} \\ e^{-i(\omega_1+\omega_2)} \\ e^{-i(\omega_1-\omega_2)} \\ 1 \end{bmatrix}.$$

This corresponds to a representation of f as

$$64f(\omega) = 5|1 - e^{-i\omega_1}|^2 + 5|1 - e^{-i\omega_2}|^2 + 4|1 - e^{-i(\omega_1+\omega_2)}|^2 + |1 - e^{-i(\omega_1-\omega_2)}|^2,$$

a sos representation of f with 4 sos generators. Observe that each of these sos generators only has 2 nonzero coefficients, which corresponds to the Cholesky factor of P only having nonzeroes on its main diagonal and last row (and in fact, this property of the Cholesky factor holds generally when using the method in Remark 1, as can be seen by inspecting the product $P = LL^*$). In this case, we see that the naive approach leads to 1 additional sos generator (and wavelet mask) with each of these sos generators having only 2 complex exponentials, while the former approach has slightly fewer sos generators (and thus wavelet masks) with the generators being more complicated. □

Example 4. [From Burt-Adelson Filters] We consider the parametrized family of lowpass filters from [2], which with our normalization is $R(\omega) = 2^{-1/2}(a + \cos(\omega) + (1-a)\cos(2\omega))$, for $a \in \mathbb{R}$. We refer to these as the Burt-Adelson (BA) masks with parameter a, and the associated filters as the BA filters with parameter a. A picture of the coset sum generated lowpass filter from the BA filter in two dimensions may be seen in Figure 2. It is easy to see that most components of Condition ($\diamondsuit$) hold automatically for these filters, with the only nontrivial one being $\alpha(1) \geq 0$. This can be shown to be equivalent to the condition

$$|a - (2^{n+1} - 3)/(2^{n+1} - 2)| \leq 2^{n/2}/(2^{n+1} - 2) \tag{12}$$

where a is the parameter of the BA filters. Thus, for each fixed $n \geq 2$, if we choose the parameter a to be $a(n) := (2^{n+1} - 3)/(2^{n+1} - 2)$, then the BA filter with parameter $a(n)$ satisfies Condition ($\diamondsuit$) for this same n. We observe that $a(n)$ is an increasing function of n with limit 1, and the BA filter with parameter 1 corresponds to nothing but the B-spline of order 2 studied in Example 1. Thus, the scaling function with parameter $a(n)$ looks Gaussian (though compactly supported) for small n, and approaches the piecewise linear B-spline as n gets larger (see also the diagrams in [2], though the parameter a used there is $a/2$ with our notation).

Let $n \geq 2$ be fixed, and suppose that the parameter a of the BA filters satisfies the condition in (12). We let

$$\begin{aligned}
v &= 2^{-n}[0, (2^n - 1)a^2 - (2^{n+1} - 3)a + 2^n - 9/4], \\
b &= -2^{-n}(2^n - 1)((2^n - 1)a^2 - (2^{n+1} - 3)a + 2^n - 9/4), \\
C &= \frac{-(1-a)^2}{4 \cdot 2^n} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\
B &= \frac{1}{4 \cdot 2^n} \begin{bmatrix} (2^{n+1} - 3)(1-a)^2 & -(1-a)^2 \\ -(1-a)^2 & -(2^{n+1} - 1)a^2 + 2(2^{n+1} - 3)a - 2(2^n - 3) \end{bmatrix}.
\end{aligned}$$

Then with the block B appearing $2^n - 1$ times on the diagonal, we have that P is given as in Equation (11). □

$$\begin{array}{ccccc} (1-a)/2 & 1/2 & \boldsymbol{a} & 1/2 & (1-a)/2 \end{array}$$

$$\Big\downarrow \mathcal{C}_2 \text{ (Coset Sum)}$$

$$\begin{array}{ccccc}
 & & (1-a)/2 & & (1-a)/2 \\
 & & 1/2 & 1/2 & \\
(1-a)/2 & 1/2 & \mathbf{3}\boldsymbol{a}-\mathbf{2} & 1/2 & (1-a)/2 \\
 & 1/2 & 1/2 & & \\
(1-a)/2 & & (1-a)/2 & &
\end{array}$$

Figure 2: Lowpass filter from the Burt-Adelson filter in Example 4 ($n = 2$).

5. Conclusion

We see that combining the coset sum method of constructing nonseparable lowpass filters from univariate ones with the idea of constructing multivariable tight wavelet frames from sum of hermitian squares representations of the function $f(\tau;\omega)$ yields many new nonseparable multidimensional tight wavelet frames. As one example of the fruitfulness of combining these ideas, consider Example 2, where we find that for any number of vanishing moments l and dimension $n \geq 2$, there is a tight wavelet frame with all highpass filters having l vanishing moments for a coset sum generated nonseparable interpolatory lowpass mask, namely the output of the coset sum method for the input Deslauriers-Dubuc mask of order $2l$.

We demonstrated a variety of methods for obtaining sos representations for nonnegative trigonometric polynomials, and demonstrated how the structured support of the trigonometric polynomials we were considering could be used to reduce the number of sos generators for these polynomials. Comparing Theorems 1 and 2, we see that in some cases, additional information about the structure of support for trigonometric polynomials can be used to find sos representations for polynomials that may not satisfy other, stronger conditions guaranteeing this existence (for the Deslauriers-Dubuc filter of order 4 in Example 2, which satisfies the hypotheses of Theorem 1, $\alpha(2) < 0$, violating Condition ($\diamondsuit$) and thus failing to satisfy the hypotheses of Theorem 2). Further exploration of the cases in which information of this kind may be leveraged to find sos representations for nonnegative trigonometric polynomials that do not clearly have a sos representation is an interesting open problem suggested by these findings.

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