Use of Quillen-Suslin Theorem for Laurent Polynomials in Wavelet Filter Bank Design

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Abstract In this chapter we give an overview of a method recently developed for designing wavelet filter banks via the Quillen-Suslin Theorem for Laurent polynomials. In this method, the Quillen-Suslin Theorem is used to transform vectors with Laurent polynomial entries to other vectors with Laurent polynomial entries so that the matrix analysis tools that were not readily available for the vectors before the transformation can now be employed. As a result, a powerful and general method for designing non-redundant wavelet filter banks is obtained. In particular, the vanishing moments of the resulting wavelet filter banks can be controlled in a very simple way, which is especially advantageous compared to other existing methods for the multi-dimensional cases.

1 Introduction

In this chapter we provide an overview of a recent method in [1] for designing non-redundant wavelet filter banks using the Quillen-Suslin Theorem for Laurent polynomials, which is a well-known result in Algebraic Geometry. The method works for any dimension but it would be the most useful for multi-dimensional cases, where the problem of designing wavelet filter banks can be quite challenging.

Wavelet representation [2], along with Fourier representation, has been one of the most commonly used data representations. Constructing 1-dimensional (1-D) wavelets is mostly well understood by now, but the situation is not the same for the multi-dimensional (multi-D) case. Taking the tensor product of 1-D functions is the most common approach, but the resulting separable wavelets have many unavoid-

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able limitations. In order to overcome these limitations, various non-tensor-based approaches for constructing multi-D wavelets have been tried, but many of these methods show limitations in various aspects as well. For example, some work only for low spatial dimensions and cannot be easily extended to higher dimensions, whereas others assume that the lowpass filters or refinable functions satisfy additional conditions such as the interpolatory condition (see, for example, [3, 4, 5, 6] and references therein). Therefore, the problem of constructing multi-D wavelets is still very challenging and calls for new ideas and insights.

Constructing wavelet filter banks is often reduced to solving an associated matrix problem with Laurent polynomial entries. Once the associated matrix problem is obtained, the wavelet filter bank design problem can be solved by using various techniques for the matrices with Laurent polynomial entries that have been developed in many different branches of mathematics. The method we look at in this chapter is based on a new way of applying the Quillen-Suslin Theorem for Laurent polynomials to the matrix problem, and it presents some advantages over the existing (both the tensor product and non-tensor-based) methods of multi-D wavelet construction: it works for any spatial dimension and for any dilation matrix, and it works without any additional assumptions, such as interpolatory condition, on the initial lowpass filters. Furthermore, it provides a simple algorithm for constructing wavelets with a prescribed number of vanishing moments.

2 Wavelet Filter Bank Design via Laurent Polynomial Matrices

Filters f are (real-valued) functions defined on the integer grids $\mathbb{Z}^n$. A filter bank (FB) consists of the analysis bank, which is a collection of, say p, filters used to analyze a given signal, and the synthesis bank, which is another (possibly different but with the same cardinality) collection of filters used to synthesize the analyzed coefficients or their modifications, depending on the application at hand, in order to get back to the original signal or its variant. We consider a special kind of FB, where one filter from each band is lowpass (i.e. $\sum_{k\in\mathbb{Z}^n} f(k) = \sqrt{q}$ where $q = |\det\Lambda|$ with dilation matrix Λ), and all the other filters are highpass (i.e. $\sum_{k\in\mathbb{Z}^n} f(k) = 0$), and we refer to such a FB as the *wavelet FB*. Only the FBs with finite impulse response filters and with the perfect reconstruction property will be considered, and in such a case we necessarily have $p \geq q$.

2.1 Polyphase Representation and Wavelet FB Design

The connection between the wavelet FB design problem and the Laurent polynomial matrix problem can be made via the polyphase decomposition [7]. Originally introduced for computationally efficient implementation of various filtering operations, the polyphase decomposition provides a way to transform filters and signals to vectors with Laurent polynomial entries, to which we refer as the *polyphase representation*. In particular, for an analysis filter h and a synthesis filter g, and for a dilation matrix Λ, the polyphase representation are given as the following Laurent polynomial vectors of length $q = |\det\Lambda| \geq 2$:

$$\mathtt{H}(z) := [H_{\nu_0}(z), \ldots, H_{\nu_{q-1}}(z)],$$

$$\mathtt{G}(z) := [G_{\nu_0}(z), \ldots, G_{\nu_{q-1}}(z)]^T,$$

respectively, where T is used for the transpose, $H_\nu(z)$ and $G_\nu(z)$ for the z-transform of the subfilters $h_\nu(k) := h(\Lambda k - \nu)$ and $g_\nu(k) := g(\Lambda k + \nu)$, respectively, and $\{\nu_0 := 0, \ldots, \nu_{q-1}\} =: \Gamma$ for a complete set of coset representatives of $\mathbb{Z}^n/\Lambda\mathbb{Z}^n$ containing 0.

In this setting, designing a FB is equivalent to finding a $p \times q$ analysis matrix $\mathtt{A}(z)$ and a $q \times p$ synthesis matrix $\mathtt{S}(z)$ with $\mathtt{S}(z)\mathtt{A}(z) = \mathtt{I}_q$. In this case, the FB is non-redundant if $p = q$, that is, if $\mathtt{A}(z)$ and $\mathtt{S}(z)$ are square. It is a wavelet FB if the first row of $\mathtt{A}(z)$ and the first column of $\mathtt{S}(z)$ are the polyphase representation of lowpass filters and all other rows of $\mathtt{A}(z)$ and all other columns of $\mathtt{S}(z)$ are the polyphase representation of highpass filters.

Understanding properties of a wavelet FB in terms of the polyphase representation is important. We recall that the filter f is lowpass (resp. highpass) if and only if $\sum_{\nu\in\Gamma} F_\nu(\mathtt{1}) = \sqrt{q}$ (resp. $\sum_{\nu\in\Gamma} F_\nu(\mathtt{1}) = 0$), where $\mathtt{1} \in \mathbb{R}^n$ is the vector of ones, and the lowpass filter f has positive accuracy if and only if $F_\nu(\mathtt{1}) = 1/\sqrt{q}$, for all $\nu \in \Gamma$ (cf. [4]). For a filter f, the number of zeros of $F(z)|_{z=e^{i\omega}}$ at $\omega \in \Gamma^* \backslash \{0\}$, where $F(z)$ is the z-transform of the filter f, is referred to as the *accuracy number* [8]. It is well known that the number of vanishing moments of each highpass filter in a non-redundant wavelet FB is at least the minimum of the accuracy numbers of the lowpass filters [9]. The number of vanishing moments is one of important criteria in determining the approximation power of a wavelet system [10].

2.2 *Quillen-Suslin Theorem and Wavelet FB Design*

A row vector of length q with Laurent polynomial entries is called *unimodular* if it has a right inverse, which is a column vector of length q. A unimodular column vector is defined similarly.

Example 1. A row vector

$$\mathtt{H}(z) = [\frac{1}{2}, \frac{1}{4}z_1^{-1} + \frac{1}{4}, \frac{1}{4}z_2^{-1} + \frac{1}{4}, \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4}] \tag{1}$$

is unimodular, because $[2,0,0,0]^T$ is a right inverse of $\mathtt{H}(z)$. In fact, there are infinitely many right inverses of $\mathtt{H}(z)$, and one of them is the column vector

$$[-\frac{1}{8}z_1^{-1} - \frac{1}{8}z_2^{-1} - \frac{1}{8}z_1^{-1}z_2^{-1} + \frac{5}{4} - \frac{1}{8}z_1 - \frac{1}{8}z_2 - \frac{1}{8}z_1z_2, \frac{1}{4} + \frac{1}{4}z_1, \frac{1}{4} + \frac{1}{4}z_2, \frac{1}{4} + \frac{1}{4}z_1z_2]^T.$$

Clearly the former is simpler, but the latter may be preferred for a wavelet FB design because the lowpass filter associated with it has larger accuracy number: it is 2, whereas the one for the former is 0. □

More generally, a matrix with Laurent polynomial entries is called a *unimodular matrix* if its maximal minors generate 1. The Quillen-Suslin Theorem (also referred to as the unimodular completion), originally conjectured by J. P. Serre [11] and proved after about 20 years [12, 13], is a well-known result in Algebraic Geometry, and it asserts that any unimodular matrix over a polynomial ring can be completed to an invertible square matrix. This result, together with its generalization to Laurent polynomial ring [14] and their constructive and algorithmic proofs [15, 16, 17], has been used in various other disciplines including Signal Processing as well [18, 19]. The following special case of the unimodular completion over Laurent polynomial rings is used for the wavelet FB design method we look at in this chapter.

Theorem 1 (Quillen-Suslin Theorem for Laurent polynomials [14]). *Let* $\mathtt{D}(z)$ *be a unimodular column vector of length q with Laurent polynomial entries. Then there exists an invertible* $q \times q$ *matrix* $\mathtt{M}(z)$ *with Laurent polynomial entries such that* $\mathtt{M}(z)\mathtt{D}(z) = [1, 0, \ldots, 0]^T$.

Although the above result can be useful in designing non-redundant wavelet FBs (cf. [9]), there are still some important questions remained to be answered. For example, obtaining a pair of lowpass filters with a prescribed number of accuracy is a key step in such an approach, but this may not be straightforward to do so, especially in multi-D cases, as we illustrate below for the 2-dimensional case.

Example 2. When $n = 2$, the lowpass filter associated with the linear box spline has accuracy 2, and its polyphase representation is given as $\mathtt{H}(z)$ in (1) and thus, as we saw in Example 1, it has a right inverse $[2, 0, 0, 0]^T$. But the lowpass filter associated with $[2, 0, 0, 0]^T$ has 0 accuracy and, as a result, it cannot be a lowpass filter for a wavelet FB. Gröbner bases techniques ([20, 21, 22]) can be used to give the most general form of the right inverse for $\mathtt{H}(z)$:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}u_1(z) \begin{bmatrix} z_1^{-1}+1 \\ -2 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}u_2(z) \begin{bmatrix} z_2^{-1}+1 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{2}u_3(z) \begin{bmatrix} z_1^{-1}z_2^{-1}+1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

(implemented via the Maple package *QuillenSuslin* by Anna Fabiańska), where $u_1(z), u_2(z), u_3(z)$ are any Laurent polynomials that are used as parameters. To find a right inverse of $\mathtt{H}(z)$ with positive accuracy, one can choose specific Laurent polynomials for parameters $u_1(z), u_2(z), u_3(z)$, which is usually done by fixing the total degree of Laurent polynomials and then increasing the total degree if needed [23, 24, 25]. However, this approach may not be the best strategy, especially if one looks for a right inverse for which the associated lowpass filter is supported in a non-rectangular region. □

3 New Quillen-Suslin based Method for Designing Wavelet FBs

In this section we discuss the main ingredients of the theory and algorithms in the new Quillen-Suslin Theorem based method for designing wavelet FBs presented in [1], and start our discussion by pointing out some motivation for the theory.

3.1 Motivation for the theory

For any lowpass filters h and g used for analysis and synthesis, respectively, their polyphase representation $\mathtt{H}(z)$ and $\mathtt{G}(z)$ satisfy the following simple matrix identity:

$$\begin{bmatrix} \mathtt{G}(z) & \mathtt{I}_q \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z) \end{bmatrix} = \mathtt{I}_q.$$

In fact, the above identity can be understood as a matrix-based interpretation of Laplacian pyramid (LP) algorithms [26], which is widely used in Signal Processing [27, 28, 29]. However, this matrix identity alone does not give a wavelet FB, because the filters associated with the column vectors of the matrix $\mathtt{I}_q$ in the synthesis matrix $\begin{bmatrix} \mathtt{G}(z) & \mathtt{I}_q \end{bmatrix}$ are not highpass, even if the lowpass filters h and g are chosen to have positive accuracy. If the lowpass filters have positive accuracy and they are biorthogonal, i.e. $\mathtt{H}(z)\mathtt{G}(z) = 1$, then another synthesis matrix $\begin{bmatrix} \mathtt{G}(z) & \mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z) \end{bmatrix}$ is available, and its use leads to the construction of wavelet FBs, as studied in [30, 31]. Actually, the most general LP synthesis matrix is known and it is

$$\mathtt{S}_{LP}(z) := \begin{bmatrix} \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z)) & \mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z) \end{bmatrix},$$

where $\mathtt{F}(z)$ is any column vector of length q [32].

Another approach to design wavelet FBs based on LP algorithms is studied in [4] for the case including when the lowpass filter h satisfies the interpolatory condition. A lowpass filter is *interpolatory* if the first component of its polyphase representation is constant, and in such a case the constant is necessarily $1/\sqrt{q}$, where $q = |\det \Lambda|$ for the dilation matrix Λ (cf. [4]). Suppose that h is interpolatory with positive accuracy. Since in this case, for any column vector $\mathtt{G}(z)$ of length q, the second row of the analysis matrix $\mathtt{A}_{LP}(z) := \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z) \end{bmatrix}$ can be written in terms of the rest rows of the matrix, we have the following identity

$$\begin{bmatrix} 1 & 0 \\ \sqrt{q}(1 - \mathtt{H}(z)\mathtt{G}(z)) & -\sqrt{q}\,\widetilde{\mathtt{H}}(z) \\ 0 & \mathtt{I}_{q-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathtt{I}_{q-1} \end{bmatrix} \mathtt{A}_{LP}(z) = \mathtt{A}_{LP}(z),$$

which in turn gives

$$\mathtt{I}_q \;=\; \mathtt{S}_{LP}(z)\mathtt{A}_{LP}(z)$$

$$
\begin{aligned}
&= \left(\mathtt{S}_{LP}(z) \begin{bmatrix} 1 & 0 \\ \sqrt{q}(1-\mathtt{H}(z)\mathtt{G}(z)) & -\sqrt{q}\,\widetilde{\mathtt{H}}(z) \\ 0 & \mathtt{I}_{q-1} \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathtt{I}_{q-1} \end{bmatrix} \mathtt{A}_{LP}(z) \right) \\
&= \begin{bmatrix} G_{v_0}(z) + \sqrt{q}\,(1-\mathtt{H}(z)\mathtt{G}(z)) & -\sqrt{q}\,\widetilde{\mathtt{H}}(z) \\ \widetilde{\mathtt{G}}(z) & \mathtt{I}_{q-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{q}} & \widetilde{\mathtt{H}}(z) \\ -\frac{1}{\sqrt{q}}\widetilde{\mathtt{G}}(z) & \mathtt{I}_{q-1} - \widetilde{\mathtt{G}}(z)\widetilde{\mathtt{H}}(z) \end{bmatrix} \\
&=: \mathtt{S}_{ECLP}(z)\mathtt{A}_{ECLP}(z)
\end{aligned}
$$

where $\widetilde{\mathtt{H}}(z)$ (resp. $\widetilde{\mathtt{G}}(z)$) is a subvector of $\mathtt{H}(z)$ (resp. $\mathtt{G}(z)$) obtained by removing the first entry. Therefore, as long as the lowpass filter g associated with $\mathtt{G}(z)$ has positive accuracy, we obtain a non-redundant wavelet FB whose analysis matrix is $\mathtt{A}_{ECLP}(z)$ and the synthesis matrix is $\mathtt{S}_{ECLP}(z)$ (cf. [4] for more details). In particular, the first column of $\mathtt{S}_{ECLP}(z)$, which is $\mathtt{G}(z) + [\sqrt{q}, 0, \ldots, 0]^T(1-\mathtt{H}(z)\mathtt{G}(z))$, is the polyphase representation of the synthesis lowpass filter.

3.2 Main ingredients of the theory

In the approach outlined above, the fact that the vector $\mathtt{H}(z)$ for the interpolatory filter has a *unit*[1] in the Laurent polynomial ring as one of its entry is used essentially, and it is clear that this property does not hold true for the general lowpass filter.

Let $\mathtt{H}(z)$ be any polyphase representation for an analysis lowpass filter h with positive accuracy (that is not necessarily interpolatory). Suppose that we want to design a non-redundant wavelet FB for which its analysis lowpass filter is h. Then $\mathtt{H}(z)$ is necessarily unimodular, because, being square matrices, the analysis matrix times the synthesis matrix equals to $\mathtt{I}_q$ as well, hence reading off $(1,1)$-entry of both sides in the identity guarantees the existence of a right inverse of $\mathtt{H}(z)$. Therefore we assume that the polyphase representation $\mathtt{H}(z)$ we start with is unimodular. The unimodularity of $\mathtt{H}(z)$ for the interpolatory h is trivial since $[\sqrt{q}, 0, \cdots, 0]^T$ is a right inverse of $\mathtt{H}(z)$.

From the unimodularity of $\mathtt{H}(z)$, we see that there exists a column vector $\mathtt{F}(z)$ of length q with Laurent polynomial entries such that $\mathtt{H}(z)\mathtt{F}(z) = 1$. Hence $\mathtt{F}(z)$ is unimodular as well. By Theorem 1, there exists an invertible $q \times q$ matrix $\mathtt{M}(z)$ such that $\mathtt{M}(z)\mathtt{F}(z) = [1, 0, \ldots, 0]^T$. Then $[\mathtt{M}(z)]^{-1}$ is a $q \times q$ matrix with Laurent polynomial entries, and $\mathtt{H}(z)[\mathtt{M}(z)]^{-1}$ is a left inverse of $\mathtt{M}(z)\mathtt{F}(z) = [1, 0, \ldots, 0]^T$, hence its first entry is 1, which is a unit. By letting the transformed row vector $\mathtt{H}^{\mathtt{M}}(z) := \mathtt{H}(z)[\mathtt{M}(z)]^{-1}$ play the role of $\mathtt{H}(z)$ in the interpolatory case as described in Section 3.1, for any column vector $\mathtt{G}(z)$ of length q, we get the following matrix identity:

$$
\mathtt{I}_q = \begin{bmatrix} \mathtt{G}^{\mathtt{M}}(z) + \mathtt{F}^{\mathtt{M}}(z)(1-\mathtt{H}^{\mathtt{M}}(z)\mathtt{G}^{\mathtt{M}}(z)) & \mathtt{I}_q - \mathtt{F}^{\mathtt{M}}(z)\mathtt{H}^{\mathtt{M}}(z) \end{bmatrix} \begin{bmatrix} \mathtt{H}^{\mathtt{M}}(z) \\ \mathtt{I}_q - \mathtt{G}^{\mathtt{M}}(z)\mathtt{H}^{\mathtt{M}}(z) \end{bmatrix}, \quad (2)
$$

[1] An element in a ring is called a unit if its multiplicative inverse lies in the ring.

where $\mathtt{F}^{\mathtt{M}}(z) := \mathtt{M}(z)\mathtt{F}(z)$ and $\mathtt{G}^{\mathtt{M}}(z) := \mathtt{M}(z)\mathtt{G}(z)$. The transformed polyphase representation used here can be thought of a generalization of the valid polyphase representation studied in [33].

Following the previous discussions when $\mathtt{H}(z)$ is interpolatory, because the second row of the transformed analysis matrix (the second matrix in the right-hand side of (2)) can be written in terms of the rest rows of the matrix, by inserting

$$\begin{bmatrix} 1 & 0 \\ (1-\mathtt{H}^{\mathtt{M}}(z)\mathtt{G}^{\mathtt{M}}(z)) & -\widetilde{\mathtt{H}^{\mathtt{M}}}(z) \\ 0 & \mathtt{I}_{q-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathtt{I}_{q-1} \end{bmatrix}$$

between the two matrices in the right-hand side of (2), we obtain the matrix identity

$$\begin{aligned} \mathtt{I}_q &= \begin{bmatrix} G^{\mathtt{M}}_{v_0}(z) + (1-\mathtt{H}^{\mathtt{M}}(z)\mathtt{G}^{\mathtt{M}}(z)) & -\widetilde{\mathtt{H}^{\mathtt{M}}}(z) \\ \widetilde{\mathtt{G}^{\mathtt{M}}}(z) & \mathtt{I}_{q-1} \end{bmatrix} \begin{bmatrix} 1 & \widetilde{\mathtt{H}^{\mathtt{M}}}(z) \\ -\widetilde{\mathtt{G}^{\mathtt{M}}}(z) & \mathtt{I}_{q-1} - \widetilde{\mathtt{G}^{\mathtt{M}}}(z)\widetilde{\mathtt{H}^{\mathtt{M}}}(z) \end{bmatrix} \\ &=: \mathtt{S}^{\mathtt{M}}_{ECLP}(z)\mathtt{A}^{\mathtt{M}}_{ECLP}(z), \end{aligned}$$

hence we get a non-redundant wavelet FB with the analysis matrix $\mathtt{A}^{\mathtt{M}}_{ECLP}(z)\mathtt{M}(z)$ and the synthesis matrix $[\mathtt{M}(z)]^{-1}\mathtt{S}^{\mathtt{M}}_{ECLP}(z)$, provided that the lowpass filter g associated with $\mathtt{G}(z)$ has positive accuracy. More precisely, in this wavelet FB, the polyphase representation for the synthesis lowpass filter is

$$[\mathtt{M}(z)]^{-1}\left(\mathtt{G}^{\mathtt{M}}(z) + [1,0,\ldots,0]^T(1-\mathtt{H}^{\mathtt{M}}(z)\mathtt{G}^{\mathtt{M}}(z))\right) = \mathtt{G}(z) + \mathtt{F}(z)(1-\mathtt{H}(z)\mathtt{G}(z)),$$

for the synthesis highpass filters are the 2nd through the last column vectors of $[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{M}(z)]^{-1}$, and for the analysis highpass filters are the 2nd through the last row vectors of $\mathtt{M}(z)[\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$.

Remark 1. *Although the case when $\mathtt{M}(z)$ satisfies $\mathtt{M}(z)\mathtt{F}(z) = [1,0,\ldots,0]^T$ is discussed above, all we need to run the above argument is for $\mathtt{M}(z)\mathtt{F}(z)$ to be a unimodular column vector with a unit in at least one of its components.*

Remark 2. *Unlike the classical approach in searching for a right inverse of $\mathtt{H}(z)$ for a non-redundant wavelet FB design (cf. Example 2), in the above approach, we do not need to look for a single right inverse of $\mathtt{H}(z)$ that has positive accuracy. Rather, one needs a pair of column vectors $\mathtt{F}(z)$ and $\mathtt{G}(z)$ such that $\mathtt{F}(z)$ is any right inverse of $\mathtt{H}(z)$ (with possibly no accuracy) and that $\mathtt{G}(z)$ has positive accuracy (but needs not be a right inverse of $\mathtt{H}(z)$), which is much easier to find.*

3.3 Main ingredients of the algorithms

The theory in the previous subsection provides an immediate algorithm for designing non-redundant wavelet FBs.

Algorithm 1: For a non-redundant wavelet FB from a lowpass filter.

Input: $\mathtt{H}(z)$*: unimodular polyphase representation of an analysis lowpass filter* h *with positive accuracy.*

Output: $\mathtt{D}(z)$*: polyphase representation of a synthesis lowpass filter,*
$\mathtt{J}_1(z),\ldots,\mathtt{J}_{q-1}(z)$*: polyphase representation of analysis highpass filters,*
$\mathtt{K}_1(z),\ldots,\mathtt{K}_{q-1}(z)$*: polyphase representation of synthesis highpass filters, such that, together with* $\mathtt{H}(z)$*, they form a non-redundant wavelet FB.*

Step 1: *Choose a lowpass filter* g *with positive accuracy, and let* $\mathtt{G}(z)$ *(as a column vector) be its polyphase representation.*
Step 2: *Choose a right inverse* $\mathtt{F}(z)$ *of* $\mathtt{H}(z)$*.*
Step 3: *Set* $\mathtt{D}(z) := \mathtt{G}(z)+\mathtt{F}(z)(1-\mathtt{H}(z)\mathtt{G}(z))$*.*
Step 4: *Choose an invertible* $q\times q$ *matrix* $\mathtt{M}(z)$ *such that* $\mathtt{M}(z)\mathtt{F}(z)=[1,0,\cdots,0]^T$*.*
Step 5: *Set* $\mathtt{J}_1(z),\ldots,\mathtt{J}_{q-1}(z)$ *:=* 2*nd through last rows of* $\mathtt{M}(z)[\mathtt{I}_q-\mathtt{G}(z)\mathtt{H}(z)]$*.*
Step 6: *Set* $\mathtt{K}_1(z),\ldots,\mathtt{K}_{q-1}(z)$ *:=* 2*nd through last columns of* $[\mathtt{I}_q-\mathtt{F}(z)\mathtt{H}(z)][\mathtt{M}(z)]^{-1}$*.*

Given an analysis lowpass filter h, if one is interested in getting a synthesis lowpass filter d with positive accuracy, one can stop the algorithm after **Step 3** and use $\mathtt{D}(z)$ there as its polyphase representation. In fact, it can be shown that the accuracy of the lowpass filter d is at least $\min\{\alpha_h,\alpha_g,\alpha_f+\beta_h,\alpha_f+\beta_g\}$, where f is the lowpass filter having $\mathtt{F}(z)$ as its polyphase representation, and α_x and β_x are the accuracy number and the flatness number of a lowpass filter x, respectively (see [1] for details including the definition of the flatness number of a lowpass filter).

In general α_f can be zero, and β_h, β_g can be as small as 1 (they have to be positive because h and g are lowpass filters) even if h and g have large accuracy, hence as a result, the accuracy of d can be much smaller than α_h. This situation can be improved by choosing g with large accuracy, and iterating a part of **Algorithm 1** as shown in the next algorithm. Recalling the close relation between the number of vanishing moments and the accuracy numbers of the lowpass filters for a non-redundant wavelet FB (cf. Section 2.1), the next algorithm provides a way to design wavelet FBs with large vanishing moments from a lowpass filter with large accuracy.

Algorithm 2: For a non-redundant wavelet FB with $\geq \alpha_h$ vanishing moments.

Input: $\mathtt{H}(z)$*: unimodular polyphase representation of an analysis lowpass filter* h *with accuracy* α_h*.*

Output: $\mathtt{D}(z)$*: polyphase representation of a synthesis lowpass filter,*
$\mathtt{J}_1(z),\ldots,\mathtt{J}_{q-1}(z)$*: polyphase representation of analysis highpass filters,*
$\mathtt{K}_1(z),\ldots,\mathtt{K}_{q-1}(z)$*: polyphase representation of synthesis highpass filters, such that, together with* $\mathtt{H}(z)$*, they form a non-redundant wavelet FB with highpass filters having at least* α_h *vanishing moments.*

Step 1: *Set* $Ite := 1$*.*
Step 2: *Choose a lowpass filter* g *with at least* α_h *accuracy, and let* $\mathtt{G}(z)$ *(as a column vector) be its polyphase representation.*
Step 3: *Choose a right inverse* $\mathtt{F}(z)$ *of* $\mathtt{H}(z)$*.*
Step 4: *Set* $\mathtt{D}(z) := \mathtt{G}(z)+\mathtt{F}(z)(1-\mathtt{H}(z)\mathtt{G}(z))$*.*

Step 5: *If* $\alpha_f + (Ite)\beta_h \geq \alpha_h$ *and* $\alpha_f + (Ite)\beta_g \geq \alpha_h$*, then go to* **Step 6**. *Otherwise, let* $Ite := Ite + 1$ *and* $\mathtt{F}(z) := \mathtt{D}(z)$*, and go to* **Step 4**.
Step 6: *Choose an invertible* $q \times q$ *matrix* $\mathtt{M}(z)$ *such that* $\mathtt{M}(z)\mathtt{F}(z) = [1, 0, \cdots, 0]^T$.
Step 7: *Set* $\mathtt{J}_1(z), \ldots, \mathtt{J}_{q-1}(z)$ *:= 2nd through last rows of* $\mathtt{M}(z)[\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$.
Step 8: *Set* $\mathtt{K}_1(z), \ldots, \mathtt{K}_{q-1}(z)$ *:= 2nd through last columns of* $[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{M}(z)]^{-1}$.

Because β_h and β_g are positive, each time the algorithm goes back to **Step 4** from **Step 5**, $\alpha_f + (Ite)\beta_h$ and $\alpha_f + (Ite)\beta_g$ strictly increase and they eventually satisfy the conditions $\alpha_f + (Ite)\beta_h \geq \alpha_h$ and $\alpha_f + (Ite)\beta_g \geq \alpha_h$, even if they did not initially. Therefore, by the time the algorithm reaches to **Step 6**, $\min\{\alpha_h, \alpha_g, \alpha_f + \beta_h, \alpha_f + \beta_g\} = \alpha_h$, and the accuracy number of the lowpass filter associated with $\mathtt{D}(z)$ is at least α_h.

In both algorithms, $\mathtt{G}(z)$, $\mathtt{F}(z)$, and $\mathtt{M}(z)$ need to be chosen. One can always choose $\mathtt{H}(z^{-1})^T$ as $\mathtt{G}(z)$. $\mathtt{F}(z)$ is nothing but the first column of $[\mathtt{M}(z)]^{-1}$, and $\mathtt{F}(z)$ and $\mathtt{M}(z)$ can be found by using Mathematical softwares such as Maple package *QuillenSuslin* mentioned earlier.

4 Conclusion

We presented some important ingredients of a recent method in [1] for designing non-redundant wavelet FBs, as well as some essential background material for the method including the Quillen-Suslin Theorem. The main advantage of this method compared to other existing wavelet FB design methods is the existence of a simple algorithm for designing a non-redundant wavelet FB with a prescribed number of vanishing moments.

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