Scalable filter banks

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ABSTRACT

A finite frame is said to be scalable if its vectors can be rescaled so that the resulting set of vectors is a tight frame. The theory of scalable frame has been extended to the setting of Laplacian pyramids which are based on (rectangular) paraunitary matrices whose column vectors are Laurent polynomial vectors. This is equivalent to scaling the polyphase matrices of the associated filter banks. Consequently, tight wavelet frames can be constructed by appropriately scaling the columns of these paraunitary matrices by diagonal matrices whose diagonal entries are square magnitude of Laurent polynomials. In this paper we present examples of tight wavelet frames constructed in this manner and discuss some of their properties in comparison to the (non tight) wavelet frames they arise from.

Keywords: Laplacian pyramid, filter banks, wavelets, paraunitary

1. INTRODUCTION

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In addition, for a $m \times n$ matrix A, A^T, denotes its transpose. $\mathcal{M}_{q,p}(z)$ will denote the set of all $q \times p$ matrices whose entries are Laurent polynomials in $z \in \mathbb{T}$ with real coefficients, and $\mathcal{M}_q(z) := \mathcal{M}_{q,1}(z)$ will denote the set of all column vectors of length q. In the sequel, unless specified otherwise, we assume that all the relations (such as identities, inequalities) among Laurent polynomial matrices in $\mathcal{M}_{q,p}(z)$ hold true for all $z \in \mathbb{T}$.

Given an integer $q \geq 2$, consider a nonzero column vector with Laurent polynomial entries $H_0(z), H_1(z), \ldots, H_{q-1}(z)$, denoted by

$$\mathtt{H}(z) := [H_0(z), H_1(z), \ldots, H_{q-1}(z)]^T \in \mathcal{M}_q(z).$$

To the (Laurent polynomial valued) vector $\mathtt{H}(z)$ we associate the *Laplacian pyramid based Laurent polynomial (LP^2) matrix* $\Phi_{\mathtt{H}}(z)$ defined by

$$\Phi_{\mathtt{H}}(z) := \left[\begin{array}{cc} \mathtt{H}(z) & \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z) \end{array}\right] \in \mathcal{M}_{q\times(q+1)}(z),$$

where $\mathtt{I}$ is the identity matrix and $\mathtt{H}^*(z)$ is the conjugate transpose of $\mathtt{H}(z)$, given by

$$\mathtt{H}^*(z) := \overline{\mathtt{H}(z)}^T = [\overline{H_0(z)}, \overline{H_1(z)}, \ldots, \overline{H_{q-1}(z)}] = [H_0(z^{-1}), H_1(z^{-1}), \ldots, H_{q-1}(z^{-1})].$$

We recall that for $z \in \mathbb{T}$, $z^{-1} = \overline{z}$. It follows that

$$\Phi_{\mathtt{H}}(z) \left[\begin{array}{c} \mathtt{H}^*(z) \\ \mathtt{I} \end{array}\right] = \mathtt{I}, \quad \forall z \in \mathbb{T}.$$

Consequently, $\operatorname{rank} \Phi_{\mathtt{H}}(z) = q$ for all $z \in \mathbb{T}$.

The LP^2 matrices are examples of Laurent polynomial matrices investigated in the setting of Laplacian pyramidal algorithms[1] using the so-called polyphase representation.[2] They appeared also in connection with various wavelet constructions.[3,4,5,6]

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The LP[2] matrix $\Phi_{\mathtt{H}}(z)$ is said to be *paraunitary,* if

$$\Phi_{\mathtt{H}}(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}.$$

The importance of paraunitary LP[2] matrices in the theory of tight filter banks[3,4] can be seen as follows. A *filter bank*[2] satisfying the perfect reconstruction property can be constructed from any pair of matrices $\mathtt{A}(z) \in \mathcal{M}_{q\times p}(z)$, $\mathtt{B}(z) \in \mathcal{M}_{p\times q}(z)$ such that $\mathtt{A}(z)\mathtt{B}(z) = \mathtt{I}$. In addition, when $\mathtt{A}(z)$ is paraunitary, i.e. $\mathtt{A}(z)\mathtt{A}^*(z) = \mathtt{I}$, the pair $(\mathtt{A}(z), \mathtt{A}^*(z))$ gives rise to a *tight filter bank*. In the sequel, we are interested in choosing $\mathtt{A}(z)$ to be an LP[2] matrix $\Phi_{\mathtt{H}}(z)$ taking advantage of the fact that the latter is generated by the single filter associated with the vector $\mathtt{H}(z)$.[5,6]

The existence of a tight filter bank from a paraunitary LP[2] matrix $\Phi_{\mathtt{H}}(z)$ is equivalent to the existence of a column matrix $\mathtt{H}(z)$ such that $\mathtt{H}^*(z)\mathtt{H}(z) = 1$, that is, $\sum_{k=0}^{q-1} |H_k(z)|^2 = 1$ for all $z \in \mathbb{T}$. So column vector without constant norm cannot be associated with a paraunitary LP[2] matrix. For example, the column vector $\mathtt{H}(z) = [1, (1 + z^{-1})/2]^T/\sqrt{2}$ has a non-constant norm and hence cannot be associated with a paraunitary LP[2] matrix.

In[7] we investigated when a column vector $\mathtt{H}(z)$ such that $\mathtt{H}^*(z)\mathtt{H}(z) \neq 1$ could be modified into a new column vector $\widetilde{\mathtt{H}}(z)$ for which $\widetilde{\mathtt{H}}^*(z)\widetilde{\mathtt{H}}(z) = 1$ leading to a paraunitary LP[2] matrix $\Phi_{\widetilde{\mathtt{H}}}(z)$. In fact, the main result proved in[7] gives a characterization of all matrices $M(z)$ whose entries are Laurent polynomials such that $\Phi_{\mathtt{H}}(z)M(z)$ is paraunitary, i.e.

$$[\Phi_{\mathtt{H}}(z)M(z)][M^*(z)\Phi_{\mathtt{H}}^*(z)] = \mathtt{I}.$$

An LP[2] matrix $\Phi_{\mathtt{H}}(z)$ for which such a diagonal matrix $M(z)$ exists is referred to as *scalable.*[7] One of the applications considered in[7] is that of the construction of tight wavelet filter banks. In the special case of univariate filter banks the result relies on the Fejér-Riesz factorization Lemma.[8,9] In this paper, we summarize this construction and provide examples of tight wavelet filter banks that can be constructed with our method. In particular, we show how to transform univariate non-tight, wavelet frames into tight wavelet ones in such a way that the resulting refinable functions preserve most of the properties of the original refinable functions.

We note that the scalar version of our results have been investigated in numerical linear algebra in the context of matrix preconditioning.[10,11] In the particular setting of finite frames[12] a special case of this question was considered under the term of *scalable frames* which were introduced in[13] (see also[14,15,16]) where one is interested in the existence of nonnegative (scalar-valued) matrices D that would make a frame with synthesis (real) matrix Φ, a tight frame, i.e., one which satisfies

$$\Phi D^2 \Phi^T = I.$$

2. SCALABLE UNIVARIATE LP[2] MATRICES

We first summarize the main results of[7] concerning scaling LP[2] matrices. The first result [7, Theorem 2.1] shows that for any LP[2] matrix $\Phi_{\mathtt{H}}(z) \in \mathcal{M}_{q\times(q+1)}(z)$, there exists a diagonal matrix $B(z) \in \mathcal{M}_{q+1,q+1}(z)$ such that

$$\Phi_{\mathtt{H}}(z)B(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}. \tag{1}$$

More specifically,

THEOREM 2.1. *[7, Theorem 2.1] Let $\Phi_{\mathtt{H}}(z)$ be an LP[2] matrix associated with $\mathtt{H}(z) \in \mathcal{M}_q(z)$. Then we have*

$$\Phi_{\mathtt{H}}(z)\mathrm{diag}([2 - \mathtt{H}^*(z)\mathtt{H}(z), 1, \ldots, 1])\Phi_{\mathtt{H}}^*(z) = \mathtt{I}.$$

Theorem 2.1 gives a sufficient condition for the diagonal matrix $B(z)$ to satisfy the identity in (1). However, the diagonal matrix $B(z)$ does not necessarily have the form given by the Theorem. Indeed, consider the case where $\mathtt{H}(z) = [0, 1]^T$, and

$$\Phi_{\mathtt{H}}(z) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We notice that if we take $B(z) = \mathrm{diag}([1, 1, 0])$, then we get the desired property, $\Phi_{\mathtt{H}}(z)B(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}$, but this $B(z)$ is not of the form in Theorem 2.1. We also notice that the LP[2] matrix $\Phi_{\mathtt{H}}(z)$ in this case is actually paraunitary, hence another possible choice for $B(z)$ is $\mathtt{I} = \mathrm{diag}([1,1,1])$, which is of the form in Theorem 2.1. In fact, any $B(z)$ of the form $B(z) = \mathrm{diag}([1,1,c])$, $c \in \mathbb{R}$, satisfies $\Phi_{\mathtt{H}}(z)B(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}$ for this example.

The following result clarifies the cases in which the solution given by Theorem 2.1 is unique.

THEOREM 2.2. *[7, Theorem 2.4] Let* $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]^T \in \mathcal{M}_q(z)$*, and let* $\Phi_{\mathtt{H}}(z)$ *be the associated LP*^{2} *matrix. Suppose that* $B(z) \in \mathcal{M}_{(q+1)\times(q+1)}(z)$ *is diagonal satisfying* $\Phi_{\mathtt{H}}(z)B(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}$*. Then* $B(z) = \mathrm{diag}([2 - \mathtt{H}^*(z)\mathtt{H}(z), 1, \ldots, 1])$ *for* $z \in \mathbb{T} \setminus S_{\mathtt{H}}$*, where the set* $S_{\mathtt{H}} \subset \mathbb{T}$ *is defined as*

$$S_{\mathtt{H}} := \{z \in \mathbb{T} : H_0(z)\overline{H_1(z)} = 0 \text{ or } 1 - |H_0(z)|^2 - |H_1(z)|^2 = 0\}$$

if $q = 2$*, and as*

$$S_{\mathtt{H}} := \{z \in \mathbb{T} : H_{k-1}(z)\overline{H_{i+k-1}(z)} = 0, \text{ for some } k = 1, \ldots, q-1,\ i = 1, \ldots, q-k\}$$

if $q \geq 3$*.*

3. UNIVARIATE TIGHT WAVELET FILTER BANKS

3.1 The theory

As an application of the results presented in Section 2 we present a new method for constructing univariate tight wavelet filter banks for any dilation parameter $\lambda \geq 2$. We first review a few basic facts on wavelets, wavelet filter banks and their polyphase representations. More details can be found, for example, in.[2,5,17]

A filter $h : \mathbb{Z} \to \mathbb{R}$ is called *lowpass* if $\sum_{k\in\mathbb{Z}} h(k) = \sqrt{\lambda}$, and *highpass* if $\sum_{k\in\mathbb{Z}} h(k) = 0$.

The z-transform of a filter h is defined as $H(z) := \sum_{k\in\mathbb{Z}} h(k)z^{-k}$. A Laurent polynomial column vector $\mathtt{H}(z) \in \mathcal{M}_q(z)$ is called the (synthesis) *polyphase representation* of a filter h if

$$\mathtt{H}(z) = [H_{\nu_0}(z), H_{\nu_1}(z), \ldots, H_{\nu_{q-1}}(z)]^T,$$

where $H_\nu(z)$ is the z-transform of the filter h_ν defined as $h_\nu(k) = h(\lambda k + \nu)$, $k \in \mathbb{Z}$.

Let h be a lowpass filter with positive accuracy, and let $\mathtt{H}(z) \in \mathcal{M}_q(z)$ be its polyphase representation. Suppose that there exists a Laurent polynomial $m_{\mathtt{H}}(z)$ such that $2 - \mathtt{H}^*(z)\mathtt{H}(z) = |m_{\mathtt{H}}(z)|^2$. Then, by Theorem 2.1 we see that

$$\Phi_{\mathtt{H}}(z)\mathrm{diag}([m_{\mathtt{H}}(z), 1, \ldots, 1]) = \begin{bmatrix} m_{\mathtt{H}}(z)\mathtt{H}(z) & \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z) \end{bmatrix}$$

is paraunitary, i.e. $\Phi_{\mathtt{H}}(z)$ is scalable (cf. Section 1).

As discussed in Section 1, the LP^2 matrix $\Phi_{\mathtt{H}}(z)$ is paraunitary if and only if $\mathtt{H}^*(z)\mathtt{H}(z) = 1, \forall z \in \mathbb{T}$. Therefore, when $\Phi_{\mathtt{H}}(z)$ itself is not paraunitary, scaling it as above can result in transforming a non-paraunitary matrix $\Phi_{\mathtt{H}}(z)$ into a paraunitary matrix $\Phi_{\mathtt{H}}(z)\mathrm{diag}([m_{\mathtt{H}}(z), 1, \ldots, 1])$. In fact, such a scaling is special in the sense that it modifies only the first column of $\Phi_{\mathtt{H}}(z)$, from $\mathtt{H}(z)$ to $m_{\mathtt{H}}(z)\mathtt{H}(z)$, while keeping all the other columns intact.

From the ongoing discussions, the construction of tight wavelet frames hinges on the existence of a Laurent polynomial $m_{\mathtt{H}}(z)$ such that $2 - \mathtt{H}^*(z)\mathtt{H}(z) = |m_{\mathtt{H}}(z)|^2$. So it is necessary that $2 - \mathtt{H}^*(z)\mathtt{H}(z) \geq 0$, for all $z \in \mathbb{T}$ which is equivalent to $\mathtt{H}^*(z)\mathtt{H}(z) \leq 2, \forall z \in \mathbb{T}$. It might suffice to rewrite $\mathtt{H}^*(z)\mathtt{H}(z)$ in terms of the mask τ: confer [7, Lemma 3.1]. When one has checked that $2 - \mathtt{H}^*(z)\mathtt{H}(z) \geq 0$, for all $z \in \mathbb{T}$, then one can appeal to the well-known Fejér-Riesz lemma[8,9] to prove the following result:

THEOREM 3.1. *[7, Theorem 3.3] Let* h *be a 1-D lowpass filter with positive accuracy and dilation* $\lambda \geq 2$*, and let* $\mathtt{H}(z)$ *be its polyphase representation. Suppose* $2 - \mathtt{H}^*(z)\mathtt{H}(z) > 0$*,* $\forall z \in \mathbb{T}$*. Then there is a polynomial* $m_{\mathtt{H}}(z)$ *such that* $[m_{\mathtt{H}}(z)\mathtt{H}(z), \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z)]$ *gives rise to a tight wavelet filter bank whose lowpass filter* $\widetilde{h}$ *is associated with* $m_{\mathtt{H}}(z)\mathtt{H}(z)$ *and has the same accuracy as* h*. Furthermore, if the support of* h *is contained in* $\{0, 1, \ldots, s\}$*, then the support of* $\widetilde{h}$ *is contained in* $\{0, 1, \ldots, 2s\}$*.*

3.2 Examples

In Burt and Adelson's original LP paper,[1] the tensor product of 1-D lowpass filter $[1/4-a/2, 1/4, a, 1/4, 1/4-a/2]$ is used, with the parameter a ranging over $\{0.3, 0.4, 0.5, 0.6\}$. In this subsection we apply our new construction method to these 1-D Burt-Adelson filters to obtain tight filter banks. We let $h := [1/4-a/2, 1/4, a, 1/4, 1/4-a/2]$ be the 1-D Burt-Adelson lowpass filter. We initially consider any real number for the parameter a, but will soon give an admissible range for a. Then, the associated z-transform $H(z)$ and refinement mask τ are given as, respectively,

$$H(z) = \sqrt{2}\left(\frac{1}{4} - \frac{a}{2}\right)\left(z^{-2} + z^2\right) + \frac{\sqrt{2}}{4}(z^{-1} + z) + \sqrt{2}a, \quad z \in \mathbb{T},$$

$$\tau(\omega) = \left(\frac{1}{2} - a\right)\cos 2\omega + \frac{1}{2}\cos\omega + a, \quad \omega \in [-\pi, \pi],$$

and the components of the polyphase representation $\mathtt{H}(z) = [H_0(z), H_1(z)]^T$, $z \in \mathbb{T}$, are given as

$$H_0(z) = \sqrt{2}\left(\frac{1}{4} - \frac{a}{2}\right)\left(z^{-1} + z\right) + \sqrt{2}a, \quad H_1(z) = \frac{\sqrt{2}}{4}(1 + z).$$

By observing that

$$\tau(\omega) = (4 - 8a)\cos^4\frac{\omega}{2} + (-3 + 8a)\cos^2\frac{\omega}{2},$$

it is easy to see that the accuracy of the refinement mask τ (or the lowpass filter h) is 4 if $a = 3/8$, and 2 if $a \neq 3/8$. Furthermore, the filter h is a three-tap filter if $a = 0.5$, and a five-tap filter if $a \neq 0.5$.

Since we have

$$2 - \mathtt{H}^*(e^{i\omega})\mathtt{H}(e^{i\omega}) = -2\left(\frac{1}{2} - a\right)^2\cos^2\omega + \left(4a^2 - 2a - \frac{1}{4}\right)\cos\omega + \frac{7}{4} - 2a^2,$$

by setting $t := \cos^2\omega$, we investigate when the polynomial $f(t) := -2\left(\frac{1}{2} - a\right)^2 t^2 + \left(4a^2 - 2a - \frac{1}{4}\right)t + \frac{7}{4} - 2a^2$ satisfies the condition

$$\max\{f(t) : t \in [-1, 1]\} = \max\{f(1), f(-1)\} = \max\{1, -8a^2 + 4a + 3/2\} > 0.$$

Therefore, we see that long as $-1/4 < a < 3/4$, the condition $2 - \mathtt{H}^*(e^{i\omega})\mathtt{H}(e^{i\omega}) > 0$ is satisfied, hence our construction method outlined in the previous subsection can be applied. In this case, from Fejér-Riesz factorization Lemma, we know that there exists $m_{\mathtt{H}}(z) = \alpha z^{-1} + \beta + \gamma z$, with $\alpha, \beta, \gamma \in \mathbb{R}$, such that $2 - \mathtt{H}^*(e^{i\omega})\mathtt{H}(e^{i\omega}) = |m_{\mathtt{H}}(e^{i\omega})|^2$. By expanding $|m_{\mathtt{H}}(e^{i\omega})|^2$ and comparing the terms in each side, we obtain

$$\alpha = \frac{1 + b + 2c}{4}, \quad \beta = \frac{1 - b}{2}, \quad \gamma = \frac{1 + b - 2c}{4},$$

where

$$b = \pm\sqrt{-8a^2 + 4a + 3/2}, \quad c = \pm\sqrt{(1 + b)^2/4 + 2(1/2 - a)^2}.$$

Hence our construction method provides a new refinement mask

$$\widetilde{\tau}(\omega) = \tau(\omega)m_{\mathtt{H}}(e^{2i\omega}) = \left(\left(\frac{1}{4} - \frac{a}{2}\right)e^{-2i\omega} + \frac{1}{4}e^{-i\omega} + a + \frac{1}{4}e^{i\omega} + \left(\frac{1}{4} - \frac{a}{2}\right)e^{2i\omega}\right)\left(\alpha e^{-2i\omega} + \beta + \gamma e^{2i\omega}\right)$$

that gives rise to the tight wavelet filter bank.

When $a = 0.5$, the original filter h is a three-tap filter with accuracy 2, and it is associated with the centered hat function: $\phi(x) = \begin{cases} 1 + x, & \text{if } -1 \leq x \leq 0, \\ 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$ Choosing $b = \sqrt{3/2}$ and $c = -(2 + \sqrt{6})/4$ gives $\alpha = 0, \beta = (2 - \sqrt{6})/4, \gamma = (2 + \sqrt{6})/4$, hence we get

$$\widetilde{\tau}(\omega) = \left(\frac{1}{4}e^{-i\omega} + \frac{1}{2} + \frac{1}{4}e^{i\omega}\right)\left(\frac{2 - \sqrt{6}}{4} + \frac{2 + \sqrt{6}}{4}e^{2i\omega}\right) = \frac{2 - \sqrt{6}}{16}e^{-i\omega} + \frac{2 - \sqrt{6}}{8} + \frac{1}{4}e^{i\omega} + \frac{2 - \sqrt{6}}{8}e^{2i\omega} + \frac{2 - \sqrt{6}}{16}e^{3i\omega},$$

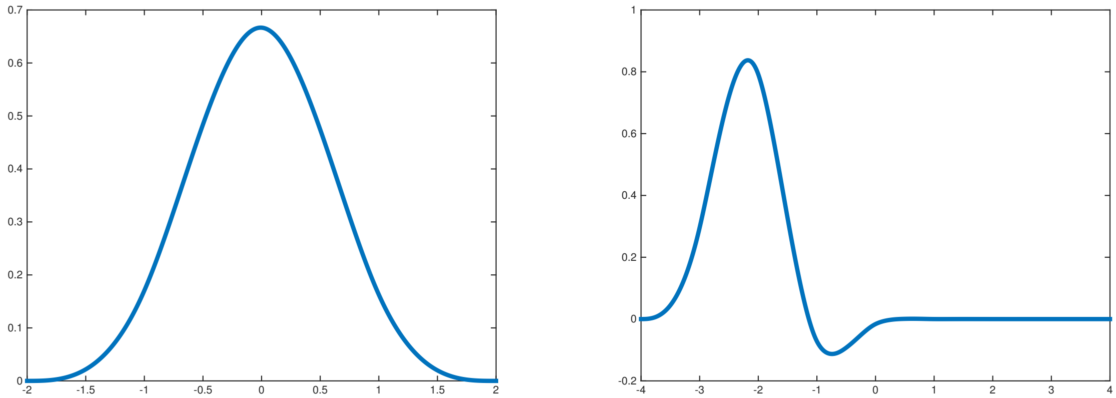

Figure 1. The original (ϕ, left) and the new ($\widetilde{\phi}$, right) refinable functions for $a = 3/8$.

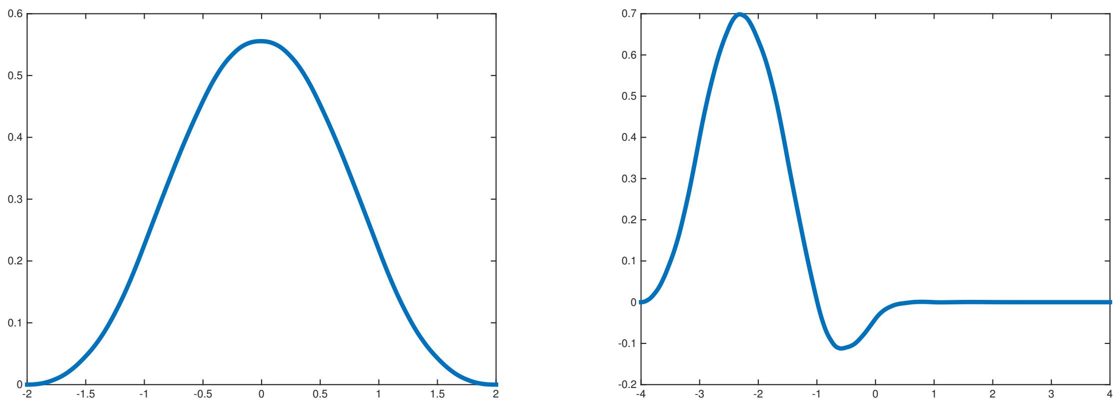

Figure 2. The original (ϕ, left) and the new ($\widetilde{\phi}$, right) refinable functions for $a = 0.3$.

whose accuracy 2 as well, by Theorem 3.1. The associated new refinable function $\widetilde{\phi}$ (with support $[-3, 1]$) is the same as the new refinable function (with support $[0, 4]$) studied in Example 1 of,[7] up to integer translation, hence we omit the graph of $\widetilde{\phi}$ in this paper. The length of support of $\widetilde{\phi}$ is twice of that of ϕ, as can be read from Theorem 3.1. Choosing other signs for b and c produces exactly the same refinable function $\widetilde{\phi}$, up to integer translation and symmetry with respect to vertical lines.

When $a = 3/8$, the original filter h is a five-tap filter with accuracy 4, and by choosing $b = \sqrt{15/8}$ and $c = -\sqrt{(1 + \sqrt{15/8})^2/4 + 2(1/2 - 3/8)^2}$, we get a new refinement mask $\widetilde{\tau}$ with accuracy 4. The graph of the new refinable function $\widetilde{\phi}$ together with the graph of the original refinable function ϕ is depicted in Fig. 1.

When $a = 0.3$ and 0.6, respectively, h is a five-tap filter with accuracy 2, and it is associated refinable function ϕ with support $[-2, 2]$, hence, by choosing $b = \sqrt{-8a^2 + 4a + 3/2}$ and $c = -\sqrt{(1 + b)^2/4 + 2(1/2 - a)^2}$, we get a new refinement mask $\widetilde{\tau}$ with accuracy 2 whose associated refinable function $\widetilde{\phi}$ is supported on $[-4, 4]$. The graphs of ϕ and $\widetilde{\phi}$ are shown in Fig. 2 and Fig. 3, respectively.

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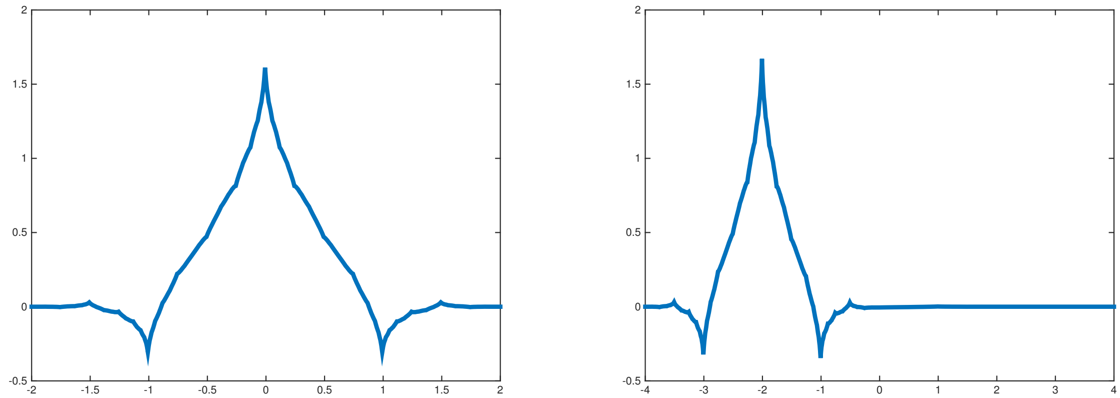

Figure 3. The original (ϕ, left) and the new ($\widetilde{\phi}$, right) refinable functions for $a = 0.6$.

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